PS #3 — Gaussian elimination and Cholesky factorization

Due: 2/5/25, 11:59 PM *Instructor: Jonathan Lindbloom*

For this problem set, you must complete at least 8 problems, *including* the required problems (*).

Problem 1. Let

$$\mathbf{N} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & l_{43} & 1 & \\ & & l_{53} & & 1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -l_{43} & 1 & \\ & & -l_{53} & & 1 \end{bmatrix}, \tag{1}$$

where l_{43} and l_{53} are arbitrary real numbers. Show that $\mathbf{NM} = \mathbf{I}$ and $\mathbf{MN} = \mathbf{I}$, i.e., that $\mathbf{M}$ is the inverse of $\mathbf{N}$.

Problem 2. Let

$$\mathbf{M}_1^{-1} = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & & 1 & & \\ l_{41} & & & 1 & \\ l_{51} & & & & 1 \end{bmatrix}, \quad \mathbf{M}_2^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & l_{32} & 1 & & \\ & l_{42} & & 1 & \\ & l_{52} & & & 1 \end{bmatrix}, \quad \mathbf{M}_3^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & l_{43} & 1 & \\ & & l_{53} & & 1 \end{bmatrix}, \quad \mathbf{M}_4^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & l_{54} & 1 \end{bmatrix} \tag{2}$$

where the l_{ij} are arbitrary real numbers. By successively computing $\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}$, $(\mathbf{M}_1^{-1}\mathbf{M}_2^{-1})\mathbf{M}_3^{-1}$, and $((\mathbf{M}_1^{-1}\mathbf{M}_2^{-1})\mathbf{M}_3^{-1})\mathbf{M}_4^{-1}$, show that

$$\mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{M}_3^{-1}\mathbf{M}_4^{-1} = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ l_{41} & l_{42} & l_{43} & 1 & \\ l_{51} & l_{52} & l_{53} & l_{54} & 1 \end{bmatrix}. \tag{3}$$

Problem 3*. GEPP applied to an invertible matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ produces a factorization

$$\mathbf{M}_2\mathbf{P}_2\mathbf{M}_1\mathbf{P}_1\mathbf{A} = \mathbf{U} \tag{4}$$

where $\mathbf{L}$ is unit lower triangular, $\mathbf{U}$ is upper triangular, and $\mathbf{P}_1$ and $\mathbf{P}_2$ are permutation matrices. Recall that $\mathbf{M}_1$ and $\mathbf{M}_2$ are elementary lower triangular matrices of the form

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}, \tag{5}$$

and that $\mathbf{P}_1$ may permute any rows while $\mathbf{P}_2$ may only permute the second and third rows.

Part (a): Explain why the factorization can be written as

$$\tilde{\mathbf{M}}_2\tilde{\mathbf{M}}_1\mathbf{PA} = \mathbf{U} \tag{6}$$

where $\tilde{\mathbf{M}}_2 = \mathbf{M}_2$, $\tilde{\mathbf{M}}_1 = \mathbf{P}_2\mathbf{M}_1\mathbf{P}_2$, $\mathbf{P} = \mathbf{P}_2\mathbf{P}_1$.

Part (b): Note that there are two possibilities for the permutation matrix $\mathbf{P}_2$. For both cases, explicitly compute the matrix $\tilde{\mathbf{M}}_1$.

Part (c): For both possibilities of $\mathbf{P}_2$, find the unit lower triangular matrix $\mathbf{L}$ such that $\mathbf{PA} = \mathbf{LU}$.

Problem 4*. In this exercise, you will analyze the computational cost to obtain the LU decomposition via GE. You may assume that no pivoting is required. Note that the elementary lower triangular matrices can be written as $\mathbf{M}_k = \mathbf{I}_n + \mathbf{m}\mathbf{e}_k^T$ where $\mathbf{m} = [0, \ldots, 0, m_{k+1,k}, \ldots, m_{n,k}]^T \in \mathbb{R}^n$ and $\mathbf{e}_k \in \mathbb{R}^n$ is the kth vector in the standard basis.

Part (a): In the kth step of GE, the multipliers in the matrix $\mathbf{M}_k$ must be computed. How many flops does computing these multipliers cost, in terms of k?

Part (b): How many flops does it take to compute a matrix-vector product $\mathbf{M}_k\mathbf{x}$ in terms of k, when computed as efficiently as possible? Note that additions or multiplications by zeros or ones that are determined in advance by the structure of $\mathbf{M}_k$ can be ignored.

Part (c): In the kth step of GE, the elementary matrix $\mathbf{M}_k$ must be applied to $\mathbf{A}_{k-1} \in \mathbb{R}^{n\times n}$. Note that the subdiagonal entries in the first $k-1$ columns of $\mathbf{A}_{k-1}$ have already been zeroed out, so we must only consider the matrix-matrix multiplication involving the matrix formed by the last $n-k+1$ columns of $\mathbf{A}_{k-1}$. How many flops does it take to compute the matrix-matrix product $\mathbf{M}_k\mathbf{A}_{k-1} = \mathbf{A}_k$, in terms of k, when computed as efficiently as possible? Again, note that additions or multiplications by zeros or ones that are determined in advance by the structure of $\mathbf{M}_k$ can be ignored.

Part (d): Use these results to show that GE to produce the factorization $\mathbf{A} = \mathbf{LU}$ costs $\mathcal{O}(n^3)$ flops.

Problem 5* (Datta 5.13). Apply GEPP and GECP (by hand, although you may use code to check your work) to both of the following matrices:

$$\text{(i)}\ \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad \text{(ii)}\ \mathbf{A} = \begin{bmatrix} 100 & 99 & 98 \\ 98 & 55 & 11 \\ 0 & 1 & 1 \end{bmatrix} \tag{7}$$

Show all of your work. For GEPP, state all matrices in the intermediate factorization

$$\mathbf{M}_2\mathbf{P}_2\mathbf{M}_1\mathbf{P}_1\mathbf{A} = \mathbf{U} \tag{8}$$

as well as the final factorization $\mathbf{PA} = \mathbf{LU}$. For GECP, state all matrices in the intermediate factorization

$$\mathbf{M}_2\mathbf{P}_2\mathbf{M}_1\mathbf{P}_1\mathbf{A}\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{U} \tag{9}$$

as well as the final factorization $\mathbf{PAQ} = \mathbf{LU}$. Compute the growth factor in all cases.

Problem 6. Explain how the linear system $\mathbf{Ax} = \mathbf{b}$ may be solved using the $\mathbf{PAQ} = \mathbf{LU}$ decomposition produced by GECP.

Problem 7. Explain how $\log(|\det(\mathbf{A})|)$ can be computed using each of the factorizations $\mathbf{A} = \mathbf{LU}$ (as in GE), $\mathbf{PA} = \mathbf{LU}$ (as in GEPP), and $\mathbf{PAQ} = \mathbf{LU}$ (as in GECP).

Problem 8 (AG 5.23). Let $\mathbf{b} + \delta\mathbf{b}$ be a perturbation of a vector $\mathbf{b}$ ($\mathbf{b} \neq \mathbf{0}$), and let $\mathbf{x}$ and $\delta\mathbf{x}$ be such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{A}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$, where $\mathbf{A}$ is a given nonsingular matrix. Show that

$$\frac{\|\delta\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \kappa(\mathbf{A}) \frac{\|\delta\mathbf{b}\|_2}{\|\mathbf{b}\|_2} \tag{10}$$

Problem 9. This problem builds on Problem 4. A matrix $\mathbf{T} \in \mathbf{R}^{n\times n}$ is said to be *tridiagonal* if $t_{ij} = 0$ whenever $|i - j| > 1$ for $i, j = 1, \ldots, n$. A matrix $\mathbf{H} \in \mathbb{R}^{n\times n}$ is said to be *upper Hessenberg* if $h_{ij} = 0$ whenever $i > j + 1$.

Part (a): Give a detailed description of how the factorization $\mathbf{H} = \mathbf{LU}$ (assuming that no pivoting is required) can be obtained in only $\mathcal{O}(n^2)$ flops when $\mathbf{H}$ is upper hessenberg.

Part (b): Give a detailed description of how the factorization $\mathbf{T} = \mathbf{LU}$ (assuming that no pivoting is required) can be obtained in only $\mathcal{O}(n)$ flops when $\mathbf{T}$ is triangular.

Problem 10. Compute (by hand, although you may use code to check your work) the Cholesky decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{bmatrix}. \tag{11}$$

Show all of your work.

Problem 11. Let $\mathbf{A} = \mathbf{LL}^\top$ be the Cholesky factorization of a symmetric positive definite matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Part (a): Explain why the diagonal of $\mathbf{A}$ can contain no negative (or zero) entries.

Part (b): Show that

$$l_{ij}^2 \leq a_{ii}, \quad i, j = 1, \ldots, n, \tag{12}$$

i.e., the squares of the entries in any row of $\mathbf{L}$ are bounded above by the corresponding diagonal entry in $\mathbf{A}$.