

Math 5: Some Basic Math Review

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March 2007. early version. more things may be added

1 Symbols

- $<$ less than, $>$ greater than
- \ll much less than, \gg much greater than (much usually meaning at least 10 times)
- \approx approximately equal to (*e.g.* 10% error)
- \sim “of order” (*i.e.* roughly same size as, but could be off by factor 2 or 3)
- \propto proportional to (*e.g.* area of circle $A \propto r^2$, time $\propto 1/\text{speed}$ when distance held constant)
- \equiv a definition, as opposed to an equation to solve
- \Rightarrow “therefore”, “it follows that”
- $x \cdot y$ another way of writing $x \times y$ or simply xy
- $|x|$ absolute value of x , that is, x if $x > 0$ but $-x$ if $x < 0$

2 Handling large and small numbers

Use powers of ten to write in *exponent notation*, *i.e.* $10^7 = 10,000,000$, $10^{-5} = 0.00001$. So,

- Radius of Earth $\approx 6,400,000$ m = 6.4×10^6 m = 6,400 km.
- Radius of human hair ~ 0.00005 m = 5×10^{-5} m = 50 μm .
- Note common scientific abbreviations: k (kilo) = 10^3 , M (mega) = 10^6 , ; m (milli) = 10^{-3} , μ (micro) = 10^{-6} , n (nano) = 10^{-9} , p (pico) = 10^{-12} ... Can you think of common words which have these abbreviations in them? (*e.g.* kilometer).
- Rule for multiplying numbers in exponent notation is $(a \times 10^n)(b \times 10^m) = ab \times 10^{n+m}$. For dividing, use $(a \times 10^n)/(b \times 10^m) = (a/b) \times 10^{n-m}$.

Significant digits: *e.g.* $1/700 = 0.001428571428 \dots = 0.001429$ (to four significant digits) or = $0.00143 = 1.43 \times 10^{-3}$ (to three significant digits). In this course, you won't need to quote answers to more than 3 or 4 significant digits; in fact it's often meaningless to do so.

PRACTICE

1. What is the cross-sectional area of the human hair mentioned above? Express your answer in both $(\mu\text{m})^2$ and in m^2 .
2. What is 1.999 expressed to 3 significant digits? How about 2.003? What is the maximum percentage error you can cause by expressing an exact number to only 3 significant digits?
3. What is the US annual military budget (\$400 billion) expressed in exponent notation?

3 Powers, logarithm, exponential

You take any number to the power of anything, *e.g.* $2^3 = 2 \times 2 \times 2 = 8$, or $3^{-2} = 1/(3 \times 3) = 1/9$, or $16^{1/2} = \sqrt{16} = 4$. [But beware non-integer powers of negative numbers \rightarrow get a complex number, yuk...] The power you take it to is called the *exponent*. Useful things:

- $a^n \cdot a^m = a^{n+m}$

- $(a^n)^m = a^{nm}$
- $a^{-n} = 1/a^n$

Logarithm (abbreviated to log) is the inverse of taking a power, *i.e.* it gives you the exponent. Logs are very common, for instance: the (semitone) scale of musical notes in Western music is logarithmic (every octave is a doubling in frequency), and sound *intensity* is measured in decibels (dB), proportional to the log of the intensity.

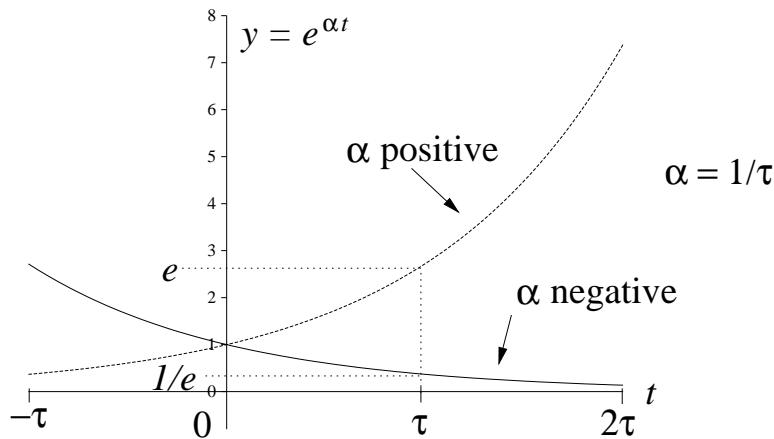
Base ten. If $y = 10^x$ then the inverse of this is $x = \log_{10} y$.

Base “e”. Defined as $e = 2.7182818 \dots$. This is the “natural” base for taking exponents and logarithms. If $y = e^x$ then the inverse of this is $x = \log y$ or less ambiguously written $\ln y$. Useful things (apply to both base e and base 10):

- $\log(ab) = \log a + \log b$. Useful since multiplication has become adding.
- $\log(1/a) = -\log a$
- $\log a^n = n \log a$

Exponential growth/decay: $y(t) = e^{\alpha t}$. (t could be time, or anything).

- $\alpha = \text{growth/decay rate} = 1/\tau$ where $\tau = \text{‘time constant’}$ (roughly, how long it is taking).
- In time τ , y gets e times bigger/smaller; in 2τ , gets e^2 times bigger/smaller; etc.
- $\alpha > 0$ gives growth (*e.g.* bacteria multiplying exponentially), $\alpha < 0$ gives decay (*e.g.* amplitude of damped oscillator once hit).



4 Dimensions, areas, volumes

We live in a three-dimensional (3d) world (we can move up/down, left/right and back/forward, giving 3 independent directions). We deal with situations with other numbers of dimensions:

- 1d: motion along a line. Amount of stuff = length (units of m). Need 1 coordinate to describe motion (*e.g.* x).
- 2d: motion on a flat plane. Amount of stuff = area (units of m^2). Need 2 coordinates to describe motion (*e.g.* x, y).
- 3d: motion in our world. Amount of stuff = volume (units of m^3). Need 3 coordinates to describe motion (*e.g.* x, y, z).

Area of a square (2d) of side length a is $A = a^2$; volume of cube (3d) of side length a is $V = a^3$. Generally, it's a^d in a d -dimensional world.

Area/volume of other simple geometric shapes:

- Circle (2d) radius r has perimeter $2\pi r$ and area πr^2 .
- Sphere (3d) radius r has surface area $4\pi r^2$ and volume $(4/3)\pi r^3$.
- Triangle (2d) has area of half base times height.

PRACTICE

1. One liter (1 l) is the volume of a cube of 10 cm side length. What is 1 l in m^3 ? How many mm^3 in a liter?
2. What is the surface area of the Earth (a sphere)?

5 Solving equations

Roman and Greek symbols a , t , ω , etc stand for *numbers*, sometimes with *units* too. *e.g.* $\omega = 3 \text{ rad s}^{-1}$, in which ω is an (angular) frequency. You *could* call this a instead, but in physics certain symbols are used by convention, and not others (follow by example).

You want to manipulate a given equation to get the an equation of the form “just the thing you want = some other stuff”. The only rules are:

- you must do the *same thing to both sides* of the equation.
- you can cancel identical things out on top and bottom of fraction, (as long as they are not equal to zero.)
- you can factorize things, *e.g.* $ab + ac = a(b + c)$.

Dimensions check. Dimensions of a quantity means how many powers of mass (M), length (L), and time (T) it has. This is common for science but we won’t emphasize it in this math course. Mostly we’ll deal with time. Valid equations must have same *dimensions* on each side. In other words a frequency cannot be equal to a time, such as $f = 4T$, because one has dimensions $1/T$, the other T, and the number 4 has no dimensions. The correct formula $f = 1/T$ has same dimensions $1/T$ on both sides. Check your dimensions agree otherwise you’ve made a mistake!

1. Solve the following equations for x : $3x + 4 = 5x - 6$, $3x^2 = 0.03$, $5e^{-x/4} = 0.5$, $a = \sqrt{4x}$, $2 + 3\log(a/x) = y$, $2^x = a$.
2. What are the dimensions of frequency? width in frequency of a peak in a frequency response curve? time constant τ ? decay rate α ? quality factor Q?
3. I have a bottle exactly twice as long as another bottle, and the same shape in every respect (*i.e.* it is an exact scale model). What is the ratio of the bottles’ surface areas? The ratio of their volumes?
4. A tuning fork’s amplitude dies with exponential decay, time constant 1 second. How long does it take for the amplitude to be 1% of the original? Does the amplitude ever reach zero?

6 Functions, graphs, slopes, areas

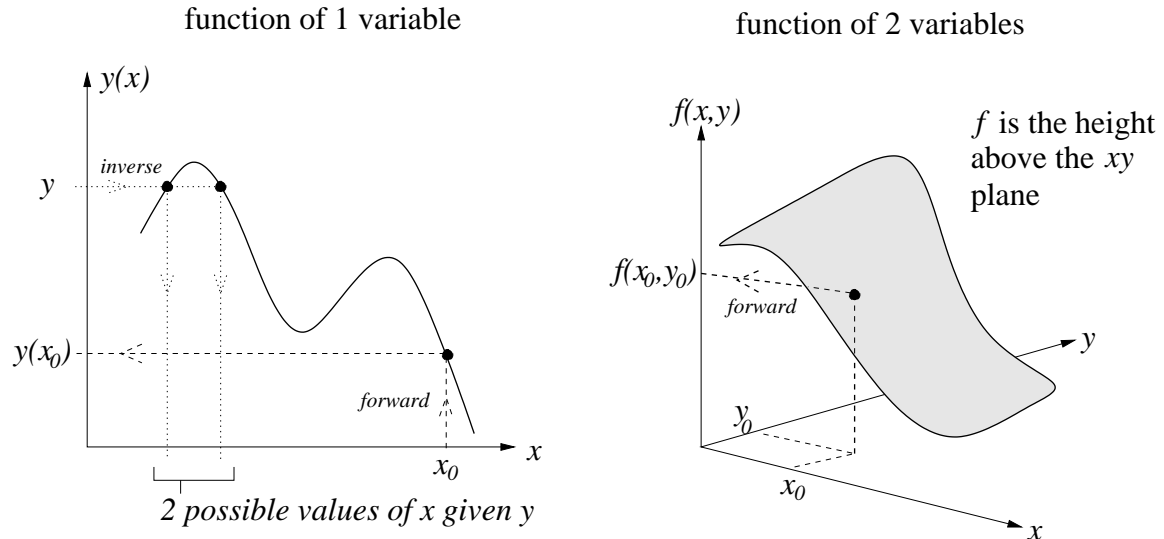
Function of one variable: If y is a function of x , it means y depends on x but not on other things. Written $y(x)$. Examples:

- position x as a function of time t is the function $x(t)$,
- height h of a landscape as a function of distance x along some road is a function $h(x)$,

- response amplitude A as a function of frequency is the function $A(f)$ or $A(\omega)$.

There is some known or unknown (but definite) form implied, *e.g.* $y(x) = 2 - 3x + x^2$, or $y(x) = 3\sin(qx)$, where q is some constant, etc.

The graph is drawn with the y -value being the height above the x axis, at each x -value. Evaluating the function is like following up from a certain point on x axis, until hits the curve, then across to y axis. The *inverse* of this means asking “what x value would give a certain y value”. It may be ambiguous, that is have many possible answers (see figure, also true for *e.g.* \sin , \cos functions).



Function of two variables: This means you need to specify *two* variables to know what value the function has. Examples:

- displacement of a bit of string d as a function of position x along the string and time t is a function $d(x, t)$,
- height h of a landscape as a function of longitude x and latitude y is a function $h(x, y)$.

Note that if you hold t constant (take a ‘snapshot’ in time), then $d(x, t)$ now becomes a $d(x)$, a function of 1 variable again. Likewise, a constant in a function of 1 variable can sometimes be considered another variable, making it a function of 2 variables (*e.g.* $y(x) = 3\sin(qx)$ with q considered a variable becomes $y(x, q) = 3\sin(qx)$).

Functions of 2 variables need to be plotted in 3d, so are hard to visualize. However, it is worth it, since we live in a world with dependence on both position and time. One other way is as a *contour plot*, looking down from above. For instance, contours of equal $h(x, y)$ would just be contours as on a topographic map.

Slope of a graph: A function $y(x)$ usually has a well-defined slope at each x value. The slope is how much y changes divided by how much x changes (these changes have to be *small*, in fact in the limit of infinitely small changes):

$$\text{slope} \equiv \frac{\text{vertical change}}{\text{horizontal change}} \equiv \frac{\Delta y}{\Delta x}.$$

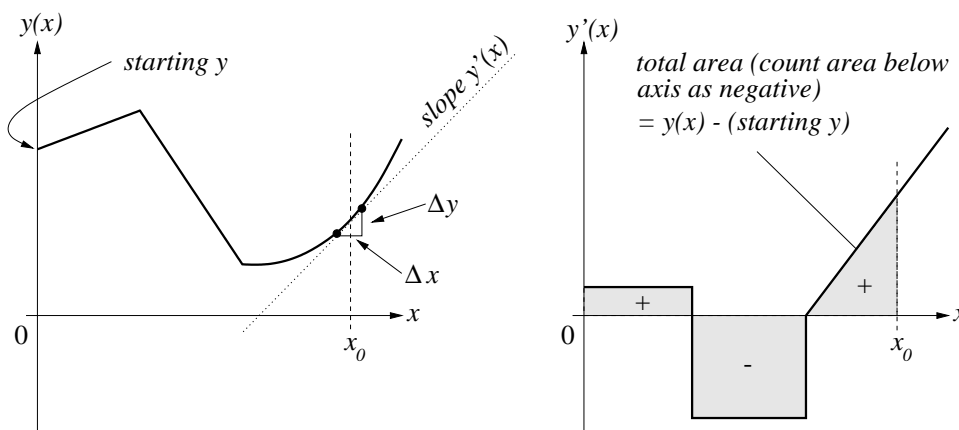
The slope is sometimes written as the new function $y'(x)$ (also called the *derivative* dy/dx). Units of slope are units of y divided by units of x . (Same for dimensions).

- The slope of position as a function of time $x(t)$ gives velocity $v(t)$.
- The slope of velocity as a function of time $v(t)$ gives acceleration $a(t)$.

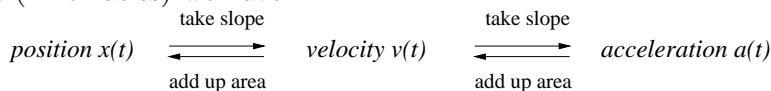
Slopes of some important functions:

- slope of a constant function is zero
- slope of a linear (straight line) function is a constant
- slope of x^2 is $2x$ (a linear function)
- slope of $\sin(x)$ is $\cos(x)$, slope of $\cos(x)$ is $-\sin(x)$. Note the angle x must be expressed in radians for these to be true!
- slope of $\sin(qx)$ is $q \cos(x)$, slope of $\cos(qx)$ is $-q \sin(x)$, where q is any constant. (Again, x must be in radians).
- slope of e^x is e^x , slope of $e^{\alpha x}$ is $\alpha e^{\alpha x}$, etc.

Area under a graph: The *inverse* of finding the slope is adding up the area under a graph. Why? Because the rate at which area is being added (*i.e.* the ‘slope of the total area function’) is just the original function itself. The only information you lose is overall vertical displacement (shifting) of the graph (*i.e.* the starting value $y(x = 0)$).



For moving objects (kinematics) we have:

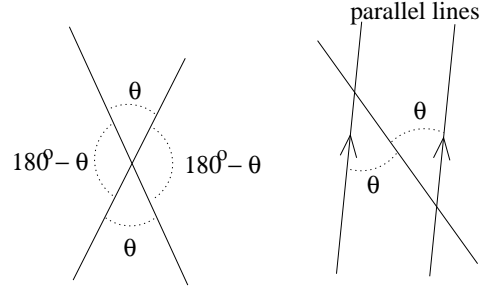
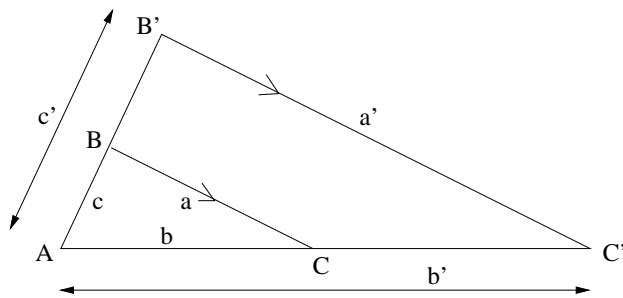


7 Geometry

Angles in a straight line add up to 180° . Angles in a triangle add to 180° .

Pythagoras’ Theorem: if a right-angle triangle has side lengths a, b, c , then this is equivalent to saying $a^2 + b^2 = c^2$. Note that c is always the *hypotenuse* (side opposite the right-angle).

Similar triangles: If a triangle has 2 angles the same as those of another triangle, then all 3 angles must be the same, and the two triangles are called *similar*. This means they have the same shape, different overall sizes. ABC and AB’C’ are similar (see figure), so side lengths have the same ratio: $a/a' = b/b' = c/c'$, and also $a/b = a'/b'$, $b/c = b'/c'$, $a/c = a'/c'$.



PRACTICE

1. Estimate how far away a person of height 1.5 m is, if they appear the same size as a 1 cm width fingertip at arm's length (70 cm). (Draw a diagram and use similar triangles).
2. What angle must the interior angles of any quadrilateral (4-sided polygon) sum to? How about a 5-sided polygon? (Chase the angles around).

8 Trigonometry

Angles can be measured in degrees ($^\circ$) or radians (rad). A complete revolution (cycle) = 360° or 2π rad. Convert using $1 \text{ rad} = (180/\pi)^\circ \approx 57.3^\circ$.

Sin, cos, tan: Given a right-angle triangle with one of the other angles θ , the *opposite* side (labelled y below, also known as the *height*) is opposite θ , and the *adjacent* side (labelled x , also known as the *base*) is next to both θ and the right-angle. The third side is the hypotenuse (labelled R). The sine (abbreviated sin), cosine (cos), and tangent (tan) functions are defined by:

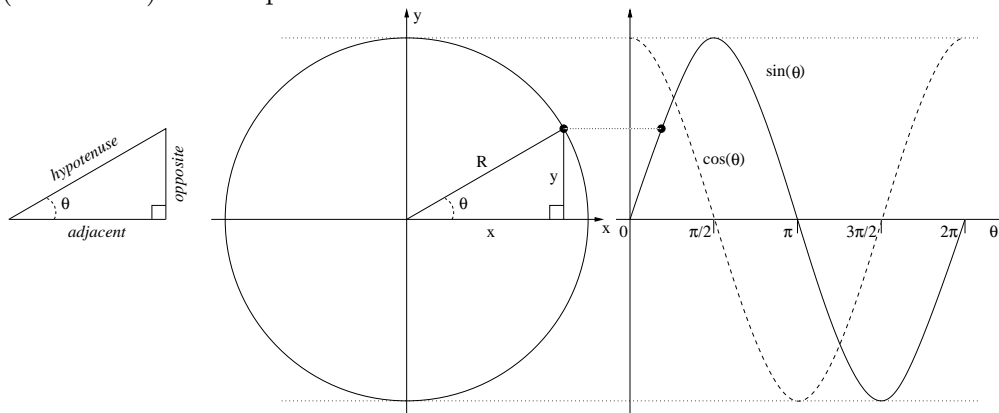
$$\sin \theta \equiv \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{R}, \quad \cos \theta \equiv \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{R}$$

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta} = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}.$$

Imagine a rod (length R), with one end fixed at the origin in a 2d plane, labelled by axes x and y . A ball sits at the other end. When the rod has angle θ to the horizontal (x axis), then

- $\sin \theta$ gives the *height* of the ball as a fraction of R ,
- $\cos \theta$ gives the *horizontal displacement* of the ball as a fraction of R , and
- $\tan \theta$ gives the *slope* of the rod.

This shows that sinusoidal motion $x(t) = R \sin(\omega t)$ is just the vertical component of uniform circular motion (*i.e.* $\theta = \omega t$) in a 2d plane.



Sin and cos always give values in the range -1 to $+1$. They repeat every cycle, so are periodic functions with period 2π .

Tan has no limits ($-\infty$ to $+\infty$), and has period π .

Note that they take as their argument an *angle* (dimensionless number). You cannot take the sin, cos or tan of something with dimensions of length, etc. Always remember whether you are working in degrees or radians, and set up your calculator similarly (test with $\sin 90^\circ = ?$)

Inverse trig functions: \sin^{-1} , \cos^{-1} , \tan^{-1} on your calculator usually give just the single solution in the angle range -90° to $+90^\circ$ ($-\pi/2$ to $+\pi/2$). However, since the functions repeat, there are an infinite number of valid solutions. *e.g.* if $\sin(\theta) = c$, then $\sin(\theta + 2\pi n) = c$ and $\sin(\pi - \theta + 2\pi n) = c$, for all integers n .

Always remember whether you are working in degrees or radians, and set up your calculator similarly (test with $\sin^{-1} 1 = ?$)

Trigonometric identities: There are loads of useful connections between sin, cos and tan of angles. Here are the ones you'll need in this course. Using Pythagoras on the above triangle gives $x^2 + y^2 = R^2$, which is the equation defining a circle in the 2d plane. Dividing by R^2 gives

$$(\sin \theta)^2 + (\cos \theta)^2 = 1, \quad \text{for any } \theta.$$

Sometimes $(\sin \theta)^2$ is abbreviated by $\sin^2 \theta$, etc.

All the following are true for *any* angles A , B :

$$\begin{aligned}\sin(-A) &= -\sin(A) \\ \cos(-A) &= \cos(A) \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin A + \sin B &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\ \cos^2 A &= \frac{1}{2}(1 + \cos 2A) \\ \sin^2 A &= \frac{1}{2}(1 - \cos 2A)\end{aligned}$$

The fifth of these is vital for understanding *beats* (Lecture 4), and how to rewrite the sum of 2 travelling waves as a standing wave (Lecture 6). The last two can be derived using the fourth, together with $\sin^2 + \cos^2 = 1$.

PRACTICE

1. What are *all* the values of θ for which $\sin \theta = 0$? $\cos \theta = 0$? $\sin \theta = -1$? $|\cos \theta| = 1$? $\sin \theta = 1/2$? $\tan \theta = 1$? Why does your calculator only give one value for inverse trigonometric functions? (\sin^{-1} etc)
2. A (huge) pie of radius 1 meter has a slice of angle 1 radian cut out of it. What is the 'arc length' (length of the curved outer edge of the slice?) What is the area of the slice?
3. A ball moves in uniform circular motion with angular velocity ω , at radius R . What is the speed of the ball? What is the maximum vertical acceleration the ball ever has? (you can do this using expressions for slope of sin and cos functions).

9 Further resources

Most high-school math books would cover this material. Also try the Schaum's series books.

Our course website Resources page has links to math teaching tools.

The best option: we have a math tutor available too!

Also come to my office hours.