Midterm Exam: Due Wednesday, May 7 at 10:00am

- You are free to use your book and notes in solving these problems.
- You may not talk to other students or discuss the problems with any other people. You also may not look up solutions to the problems online or in other books.
- Organize your solutions and write them neatly!
- Take time to explain your work and where your formulas come from. Full credit will require explanations, and it is very hard to give partial credit if you have an incorrect answer and I can not determine your line of thinking.

Problem 1: (6 points) Suppose that you roll five fair dice. What is the probability that you get a "full house", i.e. that three of the dice have one common value and the other two have a different common value?

Problem 2: (6 points) Suppose that you are playing a certain video game. When you kill a dragon, there is a 15% chance that the treasure chest it is guarding has a "Sword of Amazingness". Suppose that around the world, 30,000 people have killed the dragon 60 times each. What is probability that at least one of these 30,000 people has never received a "Sword of Amazingness" from the tresure chest?

Problem 3: Each of the parts of this problem refer to a situation where you play a sequence of games repeatedly. We assume that the probability that you win each particular game is p, and that the games are mutually independent.
(a) (6 points) Suppose you play a sequence of games repeatedly until one of you wins k times. Thus, there is a maximum of $2k - 1$ games. For each n, calculate the probability that you win the series after playing exactly n games.
(b) (2 points) Generalize part (a) as follows. Suppose that you play until either you win k games or your opponent wins ℓ games. Thus, there is a maximum of $k + \ell - 1$ games. For each n, calculate the probability that you win the series after playing exactly n games.
(b) (2 points) Recall the World Series example on the first day of class. The situation was as follows. Suppose that your team is playing in the World Series (where you and opponent play until one of you wins 4 games) and your probability of winning each game is $p = .4$. Suppose that you lose the first game. Your opponent offers you the opportunity to split the next two games (so you will be down 2-1 instead of being down 1-0). Use part (b) to calculate the probability that you win if you accept the deal and the probability that you win if you reject the deal.

Problem 4: (10 points) Suppose that you are given a finite sequence σ of H's and T's. Let p_n be the probability that in n flips of a fair coin, the sequence σ occurs consecutively at least once (for example, the sequence HHT occurs consecutively in the sequence $TTHTHHTHHH$ but the sequence TTT does not). Show that $\lim_{n \to \infty} p_n = 1$.

Problem 5: Suppose that a positive natural number n is chosen randomly according to the distribution $m(n) = 2^{-n}$. Once n is chosen, a fair die is rolled n times. Let N be the random variable which is the value of n chosen, and let S be the random variable which is the sum of the dice rolled. Calculate the following;
(a) (4 points) $P(N = 2 | S = 4)$.
(b) (4 points) $P(S = 4 | N \text{ is even})$.
(c) (4 points) The probability that the none of the dice rolled was a 6.
(d) (3 points) Are N and S independent random variables? Explain.

Problem 6: Suppose you want to model a process which gives a real number greater than or equal to 2. You have decided to model it as random variable X with density function

$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{c}{x^3} & \text{if } x \geq 2 \end{cases}$$

for some c.
(a) (4 points) Calculate the value of c which makes f a density function.
(b) (4 points) Using the value of c from part (a), calculate $E(X)$.
(c) (4 points) Using the value of c from part (a), calculate $V(X)$.

Problem 7: Suppose that you pick a point P with coordinates (X, Y) from the unit square uniformly at random. Let D be the distance from P to the nearest edge of the unit square, and let R be the product XY.
(a) (4 points) Calculate the cumulative distribution function and the density function of D.
(a) (6 points) Calculate the cumulative distribution function and the density function of R.
(a) (3 points) Are D and R independent random variables? Explain.

Problem 8: (8 points) A fair coin has probability 1/2 of getting heads. When flipping two independent fair coins, there is probability 1/4 that both are heads, 1/4 that both are tails, and 1/2 that one is heads and the other tails.

A *biased* coin has probability p of getting heads where p is a real number with $0 \leq p \leq 1$. Your friend is trying to build a pair of biased coins (not necessarily the same bias on each) such that when the two are independently flipped each of the events "Both are head", "Both are tails", and "One is heads and the other tails" have equal probability. Show that this is impossible.