**Riemann Integrability**

**Def.** A subset \( S \subset \mathbb{R} \) is a SET OF MEASURE ZERO if for every \( \varepsilon > 0 \) there exists a cover of \( S \) by a sequence of open intervals \((\alpha_i, \beta_i) \subset \mathbb{R}\) for \( i = 1, 2, 3, \ldots \)

\[
S \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)
\]

such that

\[
\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \varepsilon.
\]

**Remark:** The collection of intervals may also be finite. In this case, finite sets have measure zero.

**Theorem.** \( f : [a, b] \to \mathbb{R} \) is bounded, and \( S = \{ x \in [a, b] \mid f \text{ discontinuous at } x \} \).

If \( S \) is a set of measure zero, then \( f \) is Riemann integrable.

**REMARK.** In this case, every continuous function is integrable, as well as every \( f_n \) with finitely many discontinuities.

**Proof.** Introduce the function \( \omega : [a, b] \to \mathbb{R} \)

\[
\omega(x, \delta) = \sup \left\{ |f(y) - f(z)| \mid y, z \in (x - \delta, x + \delta) \right\}
\]

\[
\omega(x) = \inf \left\{ \omega(x, \delta) \mid \delta > 0 \right\}
\]

\( \omega(x) \) is the oscillation of \( f \) at \( x \).

**Note:** \( f \) cont. at \( x_0 \)

\[
\Rightarrow \text{ for every } \varepsilon > 0 \text{ exists } \delta > 0 \text{ s.t. } \delta \leq |x - x_0| \text{ then } |f(x_0) - f(x)| < \varepsilon.
\]

\[
\Rightarrow \text{ for every } \varepsilon > 0 \text{ exists } \delta > 0 \text{ s.t. } \omega(x_0, \delta) < 2\varepsilon
\]

\[
\Rightarrow \omega(x_0) = 0.
\]

Converse is proven in similar fashion.
Example: \( f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \)

Then \( \omega(0) = 2 \).

Now fix \( \varepsilon > 0 \). Choose \( r > 0 \) such that
\[
\frac{\varepsilon}{2} < \frac{r}{2(b-a)}.
\]

Consider the set of points \( x \) where \( f \) is discontinuous with oscillation \( \equiv r \),

\[
S_r = \{ x \in [a, b] \mid \omega(x) \equiv r \}.
\]

Claim. The set \( S_r \) is closed.

Proof of the claim: suppose \( x_0 \) is a cluster point of \( S_r \). Then for every \( \delta > 0 \) there is a point \( y \in S_r \) with \( |x_0 - y| < \delta \).

Take \( \delta' = \delta - |x_0 - y| < \delta \), then
\[
r \leq \omega(y) \leq \omega(y, \delta') \leq \omega(x, \delta)
\]
because \( (y - \delta', y + \delta') \subset (x - \delta, x + \delta) \).

Therefore \( \omega(x) \equiv r \), so \( x \in S_r \).

This proves the claim.

\( S_r \) is closed and bounded \( \Rightarrow \) compact.

By hypothesis, we can cover \( S \) and therefore also \( S_r \subset S \) by open intervals

\[
S_r \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i)
\]
with \( \sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \frac{\varepsilon}{2(b-a)} \).

Here \( B = \text{l.u.b.}\ \{ f(x) \mid x \in [a, b] \} \)
\( A = \text{g.l.b.}\ \{ f(x) \mid x \in [a, b] \} \).

Since \( S \) is compact, a finite subset suffices.
Since $S_r$ is compact, a finite subset suffices
$$S_r \subset V = (\alpha_{n_1}, \beta_{n_1}) \cup \ldots \cup (\alpha_{n_k}, \beta_{n_k})$$

wlog we can assume these are disjoint.

The complement $[a, b] \setminus V = V^c$ is closed.
For point $x \in V^c$ we have $w(x) < r$.
Because
$$w(x) = \inf \{ w(x, \delta) | \delta > 0 \}$$

it follows that for some $\delta > 0$,
$$w(x) \leq w(x, \delta) = r.$$
So now if $y, z \in (x - \delta, x + \delta)$ then $|f(y) - f(z)| < r$.
We write $I_x = (x - \frac{\delta}{2}, x + \frac{\delta}{2})$

$V^c$ is closed and bounded $\Rightarrow$ compact.
The union $\bigcup_{x \in V} I_x$ covers $V^c$ and so we
can choose a finite collection of $I_x$ that
cover $V^c$.

$$V^c \subset I_1 \cup I_2 \cup \ldots \cup I_n \ (\text{Not disjoint})$$

We now choose a partition of $[a, b]$ as follows.
Take the set of all endpoints $\{\alpha_{n_1}, \beta_{n_1}, \ldots, \alpha_{n_k}, \beta_{n_k}\}$
and all endpoints of $I_1, \ldots, I_n$ and order
them in increasing order. Ignore points in this
set if they fall outside the interval $[a, b]$,
and add $\{a, b\}$ themselves. The result will be
our partition
$$a = x_0 < x_1 < x_2 < \ldots < x_N = b.$$
There are two cases.

1. \((x_i, x_{i-1}) \subset V\). The total size of these intervals in the partition is very small
   \[ < \frac{\varepsilon}{2} \cdot \frac{1}{b-a}\]

2. \((x_{i-1}, x_i) \subset V^c\). In this case we have
   
   \((x_{i-1}, x_i) \subset I_x\) and consequently if
   
   \(y, z \in [x_{i-1}, x_i] \subset V^c\) closed interval
   
   then \(|f(y) - f(z)| < \varepsilon < \frac{\varepsilon}{2} \cdot \frac{1}{b-a}\).

For this partition, we define two step functions \(f_1, f_2 : [a, b] \rightarrow \mathbb{R}\) as follows

- If \(x = x_i\) \((i = 0, 1, \ldots, N)\) is one of the boundary points of the partition, we let
  \[ f_1(x_i) = f_2(x_i) = f(x_i). \]

- If \(x \in (x_{i-1}, x_i)\) then
  \[ f_1(x) = \inf \{ f(x) | x \in [x_{i-1}, x_i] \} \]
  \[ f_2(x) = \sup \{ f(x) | x \in [x_{i-1}, x_i] \} \]

Then clearly \(f_1(x) \leq f(x) \leq f_2(x)\) for all \(x \in [a, b]\).

Moreover, if we pick any \(t_i \in (x_{i-1}, x_i)\) then

we can calculate \(\int_a^b f_2 - f_1\). We find

\[
\int_a^b (f_2(x) - f_1(x)) \, dx = \sum_{i=1}^N (f_2(t_i) - f_1(t_i)) \cdot (x_i - x_{i-1})
\]

\[
\leq \sum_{i=1}^N (B - A) \cdot (x_i - x_{i-1}) < N \text{ from } V
\]

\[ + \sum_{(x_{i-1}, x_i) \subset V^c} \varepsilon < \varepsilon/2 \text{ from } V^c
\]

\[ \leq (B - A) \sum (x_i - x_{i-1}) < \varepsilon/2
\]
We have now satisfied the conditions of the proposition on p. 120 in the book, and it follows that \( f \) is Riemann integrable.

**Remark 1.** The converse is also true:

If \( f : [a,b] \to \mathbb{R} \) is Riemann integrable, then the set of points where \( f \) is discontinuous has measure zero, and \( f \) is bounded.

So now we know exactly which functions have a Riemann integral and which do not.

**Remark 2.** All finite sets have measure zero. Therefore all bounded functions that have no more than a finite nr. of discontinuities are Riemann integrable. Step functions are a special case.

But it is also true that all countably infinite sets have measure zero. Here is a proof.

Suppose the set \( S \) of points where \( f \) is discontinuous is countable. This means that we can label these points by an integer suffix:

\[
S = \{a_1, a_2, a_3, a_4, \ldots \}
\]

Now fix \( \varepsilon > 0 \), and let

\[
(a_i, \beta_i) = (a_1 - \frac{\varepsilon}{2i}, a_1 + \frac{\varepsilon}{2i})
\]

The union of all \((a_i, \beta_i)\) covers \(S\).
The union of all $(a_i, b_i)$ covers $S$. And the total size of these open intervals is
\[ \sum_{i=1}^{\infty} (b_i - a_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \left( \sum_{i=1}^{\infty} \frac{1}{2^i} \right) = \varepsilon. \]

It follows that the “ruler function”
\[ \varphi : [0,1] \to \mathbb{R} \]
\[ \varphi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \end{cases} \]

is Riemann integrable!!! The function $\varphi$ is discontinuous only at rational points. The set of rational numbers is countably infinite, and therefore has measure zero.

**Remark.** Things can get even worse. There exist sets of measure zero that are UNCOUNTABLY infinite. (For example: the Cantor set), and there are Riemann integrable functions with discontinuities on such a set. However, there are also functions that are NOT integrable. For example
\[ \varphi(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1 & \text{if } x \text{ rational} \end{cases} \]
\[ \varphi : [0,1] \to \mathbb{R}. \]

This function is NOWHERE continuous, and the set $S = [0,1]$ does not have measure zero, so $\varphi$ is NOT Riemann integrable.