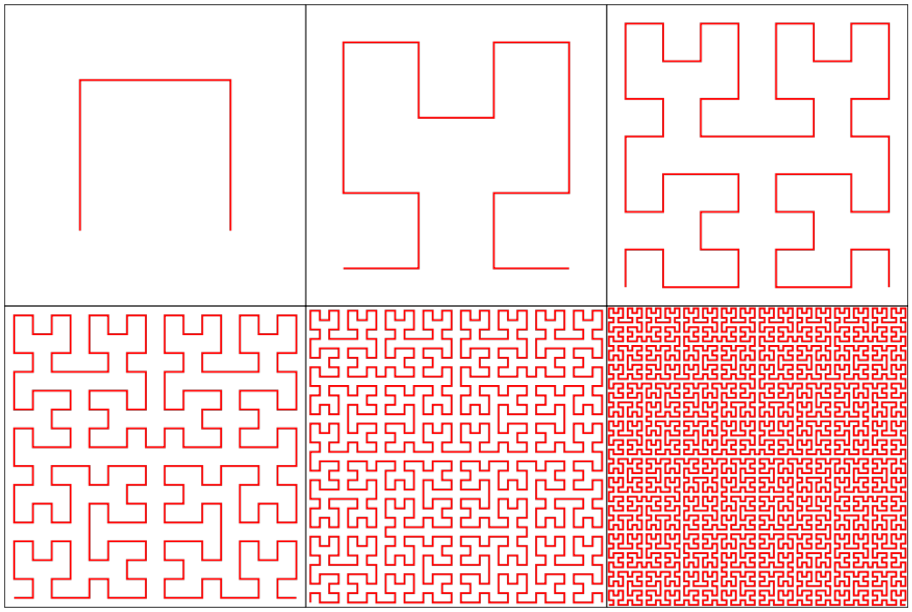
Math 63, Winter 2019

The Hilbert curve (1891).
The following picture is taken from the wikipedia entry for *Space-filling curve.*



This picture consists of six separate graphs, each depicting a curve in the plane. The curves depicted here represent the first six functions in a sequence $f_1, f_2, f_3, \ldots$ with $f_n \colon [0,1] \to \mathbb{R}^2$. Each picture shows, in red, a subset of the plane $\mathbb{R}^2$ that is the *image* $\{f_n(t) \mid t \in [0,1]\}$ for one of these functions. The particular sequence of functions depicted here was constructed by the mathematician David Hilbert in 1891, inspired by a similar idea of Guiseppe Peano from the year before.

Consider the picture in the upper left hand corner. It depicts the image of a function $f_1 \colon [0,1] \to \mathbb{R}^2$. To describe the function f_1, we divide the interval $[0,1]$ into four equal (but not disjoint) parts

$$[0,1] = [0, \frac{1}{4}] \cup [\frac{1}{4}, \frac{2}{4}] \cup [\frac{2}{4}, \frac{3}{4}] \cup [\frac{3}{4}, 1] = I_1 \cup I_2 \cup I_3 \cup I_4.$$

We also divide the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ into 4 equal parts:

$$\begin{aligned}[0,1] \times [0,1] &= [0, \frac{1}{2}] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ &= S_1 \cup S_2 \cup S_3 \cup S_4.\end{aligned}$$

Then the crucial property of the function f_1 is that it maps each interval I_p to the square S_p,

$$f_1(I_p) \subset S_p \text{ for } p = 1, 2, 3, 4.$$

Moreover, if you look at all the five pictures of f_2, f_3, f_4, f_5, f_6 that come after f_1, you may see that they have the same property:

$$f_n(I_p) \subset S_p \text{ , for } p = 1, 2, 3, 4, \text{ and for every } n \geq 1.$$

The curves f_n become more and more convoluted, but remain subordinate to this rule imposed by the division of the interval $[0, 1]$ and the square $[0, 1] \times [0, 1]$ into four equal parts.

In general, for every value of $n = 1, 2, 3, \ldots$, there exists a subdivision of the interval $[0, 1]$ into 4^n closed intervals of size $1/4^n$,

$$[0, 1] = \bigcup_{p=1}^{4^n} I_p^n.$$

The explicit formula for the interval I_p^n is

$$I_p^n = [\frac{p-1}{4^n}, \frac{p}{4^n}].$$

Likewise, the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ can be sub-divided into 4^n smaller squares with sides of length $1/2^n$. Let us assume that we label this set of squares somehow as $S_1^n, S_2^n, \ldots, S_{4^n}^n$, so that

$$[0, 1] \times [0, 1] = \bigcup_{p=1}^{4^n} S_p^n,$$

where each S_p^n is of the form

$$S_p^n = [\frac{q-1}{2^n}, \frac{q}{2^n}] \times [\frac{r-1}{2^n}, \frac{r}{2^n}],$$

for some choice of integers q, r between 1 and 2^n.

Hilbert proved (and you may take this for granted) that it is possible to label the squares S_p^n in such a way that there exists a *continuous* function $f_n\colon [0, 1] \to \mathbb{R}^2$ which satisfies

$$f_n(I_p^n) \subset S_p^n \text{ for } p = 1, 2, \ldots, 4^n.$$

Moreover, all of the functions f_m in the sequence with $m > n$ also respect this particular subdivision into 4^n parts,

$$f_m(I_p^n) \subset S_p^n, \text{ for all } m \geq n, \ p = 1, 2, \ldots, 4^n.$$

Before you try to solve the following problems, identify the squares S_p^n in the six pictures and try to understand how the properties of f_n mentioned here are reflected in the graphs.

Let $\mathcal{F}$ be the set of continuous functions $f : [0, 1] \to \mathbb{R}^2$, and let $D : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ be the metric

$$D(f, g) = \max_{t \in [0,1]} d(f(t), g(t))$$

where d is the Euclidean distance in $\mathbb{R}^2$.

1. Prove that $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $\mathcal{F}$.
2. Why does this imply that $\lim_{n\to\infty} f_n = f$ exists in $\mathcal{F}$? Why does this imply that $f : [0, 1] \to \mathbb{R}^2$ is continuous?
3. Prove that for every point $(x, y) \in [0, 1] \times [0, 1]$ there exists $t \in [0, 1]$ such that $f(t) = (x, y)$.

Remark. The limit function f is called the *Hilbert curve* and it is an example of a *space-filling curve.*