Selected Solutions for Math 63 Homework #1

1. Chapter I: #10ab.

   ANS: This is a test to see if you can parse a definition precisely. Recall that a function $f : A \to B$ is a subset $f \subset A \times B$ such that for *every* $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. If *either* A or B is empty, then so is $A \times B$. Hence there is only one subset of $A \times B$ — namely the empty set $\emptyset$. The only question is whether or not the empty set is a function.

   (a) If A is nonempty and $B = \emptyset$, then given $a \in A$, there can be no $(a, b) \in A \times B = \emptyset$, so there are no functions $f : A \to \emptyset$.

   (b) On the other hand, if $A = \emptyset$, then whether or not B is empty, the condition for all $a \in A$ there is a unique $b \in B$ such that $(a, b) \in \emptyset$ is vacuously satisfied. Hence the empty set is a function, and the only function, from $\emptyset$ to B.

2. Show that in $\mathbf{R}$, if $x \geq -1$ and $n \in \mathbf{N}$, then $(1 + x)^n \geq 1 + nx$. I suggest using induction. (You can work this problem in any ordered field if we define $(n+1) \cdot x = n \cdot x + x$ for $n \in \mathbf{N}$. But it is fine to work in $\mathbf{R}$.)

   ANS: The assertion is immediate if $n = 1$. So assume the result for $n \geq 1$. Then

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) \tag{1}$$

   which, since $1 + x \geq 0$ and the inductive hypothesis, is

$$\geq (1 + nx)(1 + x) = 1 + nx + x + nx^2 \tag{2}$$

   which, since $nx^2 \geq 0$, is

$$\geq 1 + (n + 1)x. \tag{3}$$

   This completes the proof.

3. Chapter II: #2.

   ANS: We have $(a - b) + (b - a) = (a + (-b)) + (b + (-a)) = a + (-b + b) + (-a) = a + 0 + (-a) = a + (-a) = 0$. Hence $b - a$ is the additive inverse of $a - b$ so by definition $b - a = -(a - b)$.

4. Chapter II: #3.

   ANS: Suppose that $a < b < 0$. Then $a^{-1} < 0$ and $b^{-1} < 0$ by (O 7). Then $a^{-1}b^{-1} > 0$. Since $b - a > 0$, we have $a^{-1}b^{-1}(b - a) > 0$ and $(a^{-1}b^{-1})b > (a^{-1}b^{-1})a$. Thus $\frac{1}{a} = a^{-1} > \frac{1}{b} = b^{-1}$ as claimed.

5. Chapter II: #6.

ANS: First I claim that if $a < b$, then $-b < -a$. To see this, note that $b - a > 0$ and $-1 < 0$. Therefore we proved in class that $-1(b-a) < 0$ (i.e., positive times negative is negative). But then $-b + a = -b - (-a) < 0$ and $-b < -a$ as claimed.

Now $a < y < b$ implies $0 < y - a < b - a$. But $a < x < b$ implies $-x < -a$. Then $y - x < y - a$. By transitivity, $y - x < b - a$. But the situation is symmetric in x and y so we also have $-(y - x) = x - y < b - a$. Thus $|x - y| < b - a$ as claimed.

6. Chapter II: #11.

ANS: Here we have to show that if $a > 1$, then $\{a, a^2, a^3, \dots\}$ is not bounded. Ok, suppose not. Then there is $x \in \mathbf{R}$ such that $a^n \le x$ for all $n \in \mathbf{N}$.

Next we turn to the hint. We proved earlier that if $x \ge -1$, then $(1+x)^n \ge 1 + nx$. If we let $x = \frac{1}{n}$, then

$$\Bigl(1 + \frac{1}{n}\Bigr)^n \ge 2 \quad \text{for all } n \in \mathbf{N}. \tag{$\dagger$}$$

Next I claim that

$$2^k \ge k \quad \text{for all } k \in \mathbf{N}. \tag{$\ddagger$}$$

Again, we'll use induction. The result is clear for $k = 1$. So assume that the result holds for k. Then

$$2^{k+1} \ge 2^k(2) \ge k \cdot 2 = k + k \ge k + 1.$$

Thus ($\ddagger$) holds for all k. But there is a $k > x$ (by LUB 1). Since $a - 1 > 0$, there is a $n \in \mathbf{N}$ such that $\frac{1}{n} < a - 1$ and

$$a > \bigl(1 + \frac{1}{n}\bigr).$$

But then

$$\begin{aligned} a^{kn} &> \Bigl(\Bigl(1 + \frac{1}{n}\Bigr)^n\Bigr)^k \\ &\ge 2^k \\ &\ge k \\ &> x. \end{aligned}$$

But this contradicts our choice of x. This finishes the proof.

METHOD II (WITHOUT THE HINT): Again, suppose to the contrary that $S = \{\, a^n : n \in \mathbf{N} \,\}$ is bounded above. Let $u = \text{l.u.b.}(S)$. Then $u \ge a > 1$. But $a > 1$ implies $ua > u$. If we take $k \in \mathbf{N}$ such that $\frac{1}{k} < ua - u$, then $ua - \frac{1}{k} > u$. But we also have $\frac{1}{a} < 1$ so that $\frac{1}{ak} < \frac{1}{k}$. It follows that $ua - \frac{1}{ak} < ua - \frac{1}{k} < u$. Then there is a $n \in \mathbf{N}$ such that $a^n > u - \frac{1}{ak}$. But then $a^{n+1} > ua - \frac{1}{k} > u$. But this contradicts the definition of u.

7. Chapter II: #13.

ANS: Since each S_i is nonempty and bounded above, each set has a least upper bound s_i. Define

$$S_1 + S_2 = \{ x + y : x \in S_1 \text{ and } y \in S_2 \}.$$

We are supposed to show that $\text{lub}(S_1 + S_2) = s_1 + s_2$. But if $x \in S_1$ and $y \in S_2$, then

$$x + y \leq s_1 + s_2.$$

Hence $S_1 + S_2$ is bounded above (as well as nonempty). Hence $S_1 + S_2$ at least has an least upper bound. Since $s_1 + s_2$ is an upper bound, it will suffice to see that $s_1 + s_2 - \epsilon$ is not an upper bound for any $\epsilon > 0$. But $s_1 - \epsilon/2$ can't be an upper bound for S_1. Thus there is a $t_1 \in S_1$ such that $t_1 > s_1 - \epsilon/2$. Similarly, there is a $t_2 \in S_2$ such that $t_2 > s_2 - \epsilon/2$. But now we have $t_1 + t_2 \in S_1 + S_2$ and

$$t_1 + t_2 > s_1 + s_2 - \epsilon.$$

Thus $s_1 + s_2 - \epsilon$ is not an upper bound and we're done.