Math 63: Winter 2021
Lecture 3

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Getting Started

1. We should be recording.
2. Our First Homework assignment is due Friday at 3pm via gradescope.
3. Time for some questions!

Last Time

Definition

We say that a field $\mathbf{F}$ is ordered if there is a subset $P \subset \mathbf{F} \setminus \{0\}$ such that

1. $\mathbf{F}$ is the disjoint union of P, $\{0\}$, and $-P$.
2. If $a, b \in P$, then $a + b \in P$ and $ab \in P$.

We say that $x > 0$ if $x \in P$, and that $x < y$ if $y - x \in P$. We call the pair $(\mathbf{F}, P)$, or sometimes $(F, <)$ an ordered field.

Example

Let $P = \{ \frac{a}{b} \in \mathbf{Q} : a, b \in \mathbf{N} \}$. Then $(\mathbf{Q}, P)$ is an ordered field that we've held dear to our hearts since grade school.

Not a Rational World

Remark

We know, or at least we've been told, that rational numbers do not suffice for the sort of "real world" applications we love. Assuming, as we'll show later, that $\sqrt{2}$ is not rational, the distance from $(1, 0)$ to $(0, 1)$ in the plane is not rational. The area of a circle of radius 1 is not rational. The function $f(x) = x^2 - 2$ does not cross the x-axis in a rational world.

Since back in the day, we have been told that the rational numbers are actually a subfield of a field, $\mathbf{R}$, called the field of real numbers that contains all the numbers we want such as $\sqrt{2}$, π, e, as things we haven't even thought of yet. So what makes $\mathbf{R}$ special?

Least Upper Bounds

Definition

Let $S \subset \mathbf{F}$ be a subset of an ordered field. We say that b is an upper bound for S if $s \leq b$ for all $s \in S$. We say that u is a least upper bound for S if u is an upper bound for S such that if t is any other upper bound for S then $u \leq t$. If u exists, then we write $u = \mathrm{l.u.b.}(S)$.

Example

Let $S = \{\, 1 - \frac{1}{n} : n \in \mathbf{N} \,\}$. Then $1 = \mathrm{l.u.b.}(S)$. Note that $1 \notin S$.

Example

Let $\mathbf{F} = \mathbf{Q}$ and $S = \{\, r \in \mathbf{Q} : r^2 < 2 \,\}$. Then $b = 2$ is an upper bound for S in $\mathbf{Q}$. But assuming $\sqrt{2}$ is irrational and that we can find rational numbers as close as we please to $\sqrt{2}$, then S has no least upper bound in $\mathbf{Q}$. Of course, in $\mathbf{R}$, $\mathrm{l.u.b.}(S) = \sqrt{2}$.

Complete Ordered Fields

Definition

An ordered field $\mathbf{F}$ is called complete if every nonempty set that is bounded about has a least upper bound.

Definition

We assume that there is a complete ordered field, $\mathbf{R}$, called the real numbers, containing $\mathbf{Q}$ as a subfield.

Remark (Aside)

Actually, we don't have to assume $\mathbf{Q} \subset \mathbf{R}$. We call a subset $A \subset \mathbf{R}$ inductive if $1 \in A$ and $n \in A$ implies $n+1 \in A$. Since $\bigcap\{\, A : A \text{ is inductive} \,\}$ is inductive, there is a smallest inductive set $\mathbf{N}$ in $\mathbf{R}$. We let $\mathbf{Z} = -\mathbf{N} \cup \{0\} \cup \mathbf{N}$ and define $\mathbf{Q}$ as before.

Some Real Properties

Lemma

If $x \in \mathbf{R}$, then there is a $n \in \mathbf{N}$ such that $n > x$.

Proof.

Suppose that result is false. Then there is an upper bound $x \in F$ for the set $\mathbf{N}$. Let $a = \mathrm{l.u.b.}(\mathbf{N})$. Then $a - 1$ is not an upper bound for $\mathbf{N}$. Why? Thus there is a $n \in \mathbf{N}$ such that $a - 1 < n$. But then $a < n + 1$. Since $n + 1 \in \mathbf{N}$, a is not an upper bound for $\mathbf{N}$. This is a contradiction. □

Lemma

Suppose $\epsilon > 0$ in $\mathbf{R}$. Then there is a $n \in \mathbf{N}$ such that $\frac{1}{n} < \epsilon$.

Proof.

Use the first lemma, to find $n \in \mathbf{N}$ such that $\frac{1}{\epsilon} < n$. Since $0 < a < b$ implies $\frac{1}{b} < \frac{1}{a}$ (HW), we have $\frac{1}{n} < \epsilon$. □

Getting There

Lemma

If $x \in \mathbf{R}$ then there is a $n \in \mathbf{Z}$ such that $n \leq x < n+1$.

Proof.

Let $N \in \mathbf{N}$ be such that $|x| \leq N$. Then $-N \leq x \leq N$. Since $F = \{ -N, -N+1, \dots, N \}$ is finite, we can let n be the smallest $n \in F$ such that $n \leq x$. □

Lemma

Let $x \in \mathbf{R}$ and $N \in \mathbf{N}$. Then there is a $n \in \mathbf{Z}$ such that $\frac{n}{N} \leq x < \frac{n+1}{N}$.

Proof.

Use the previous result to find $n \in \mathbf{Z}$ such that $n \leq Nx < n+1$. □

The Rationals are Dense

Proposition

Suppose $x \in \mathbf{R}$ *and* $\epsilon > 0$. *Then there is a* $r \in \mathbf{Q}$ *such that* $|x - r| < \epsilon$. *(Later, we will say that* $\mathbf{Q}$ *is dense in* $\mathbf{R}$*.)*

Proof.

We can find $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$. Then we can find $n \in \mathbf{Z}$ such that $\frac{n}{N} \leq x < \frac{n+1}{N}$. Then $0 \leq x - \frac{n}{N} < \frac{1}{N} < \epsilon$. Hence $|x - \frac{n}{N}| < \epsilon$. □

Break Time

1. Well, that was fun.
2. Time for a break.

Decimals

Definition

Let $D = \{0, 1, 2, \dots, 0\}$. If $\{a_1, \dots, a_n\} \subset D$ and $a_0 \in \mathbf{Z}$, then we define

$$a_0.a_1a_2\cdots a_n = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}. \qquad (\ddagger)$$

We call $(\ddagger)$ a finite decimal expansion.

Lemma (Rounding Off)

Let $a_0.a_1\cdots a_n$ be a finite decimal expansion. If $1 \leq m < n$, then

$$a_0.a_1\cdots a_m \leq a_0.a_1\cdots a_n < a_0.a_1\cdots a_m + \frac{1}{10^m}.$$

Remark

For example, $1.23 \leq 1.2345 < 1.23 + 0.01$. (Here, $m = 2$.)

Proof

Proof of the Lemma.

We have

$$\begin{aligned} a_0.a_1 \cdots a_m &\leq a_0.a_1 \cdots a_m \\ &= a_0.a_1 \cdots a_m + a_{m+1}10^{-(m+1)} + \cdots + a_n 10^{-n} \\ &\leq a_0.a_1 \cdots a_m + 9 \cdot 10^{-(m+1)} + \cdots + 9 \cdot 10^{-n} \end{aligned}$$

which, after adding $10^{-n} > 0$ to the right-hand side, is

$$\leq a_0.a_1 \cdots a_m + 10^{-m}.$$

□

Infinite Decimals

Given $a_0 \in \mathbf{Z}$ and a sequence $(a_n) \subset \{ 0, 1, 2, \cdots, 9 \}$, consider the set

$$S = \{ a_0.a_1 \cdots a_n : n \in \mathbf{N} \}.$$

This set is nonempty bounded aboue by $a_0.a_1 + 0.1$. Hence we can define

$$a_0.a_1 a_2 \cdots = \text{l. u. b.} \{ a_0.a_1 \cdots a_n : n \in \mathbf{N} \}.$$

We call $a_0.a_1 a_2 \cdots$ an infinite decimal expansion.

Theorem

If $x \in \mathbf{R}$, then $x = a_0.a_1 a_2 \cdots$ for some $a_0 \in \mathbf{Z}$ and sequence $(a_n) \subset \{ 0, 1, \dots, 9 \}$. We say that $a_0.a_1 a_2 \cdots$ is a decimal expansion for x.

Proof

Sketch of the Proof.

Given $m \in \mathbf{N}$, there is a $n \in \mathbf{Z}$ such that $\frac{n}{10^m} \leq x \leq \frac{n+1}{10^m}$. Thus $a_0.a_1 \cdots a_m \leq x \leq a_0.a_1 \cdots a_m + 10^{-m}$ for appropriate a_k. With more work than I want to put in here, we can see that the a_k are uniquely determined by x. Thus we can find a_{m+1} such that

$$a_0.a_1 \cdots a_m a_{m+1} \leq x \leq a_0.a_1 \cdots a_m a_{m+1} + 10^{-(m+1)}.$$

Continuing inductively, we get a sequence (a_k) such that

$$x = \text{l. u. b.}\{\, a_0.a_1 \cdots a_m : m \in \mathbf{N} \,\} = a_0.a_1 a_2 \cdots . \qquad \square$$

Caution

Remark

Our theorem implies that every real number has an infinite decimal expansion, but for example, $1.2499999\cdots = 1.2500000\cdots = 1.25$. But this is all that can go wrong.

Break Time

1. Let's take a break.

Theorem

If $a \in \mathbf{R}$ and $a > 0$, then there is a unique $y > 0$ such that $y^2 = a$. Colloquially, every positive real number has a unique positive square root.

Proof.

If $0 < x < y$, then $x^2 < y^2$. Hence if a a positive square root, then it is unique. Thus we just have to show existence.
Let $S = \{ x \geq 0 : x^2 \leq a \}$. Since $0 \in S$, S is not empty. Let $b = \max\{ a, 1 \}$. Then $b^2 = b \cdot b \geq b \cdot 1 = b \geq a$. Hence b is an upper bound for S. Let $y = \text{l. u. b.}(S)$. We want to show that $y^2 = a$.

Proof

Proof Continued.

If $b = \min\{ a, 1 \}$, then $b^2 = b \cdot b \leq b \cdot 1 = b \leq a$. Hence $b \in S$. Thus $y > 0$.

Now suppose $0 < \epsilon < y$. Then $0 < y - \epsilon < y < y + \epsilon$ and $(y-\epsilon)^2 < y^2 < (y+\epsilon)^2$. Since $y = \text{l.u.b.}(S)$, there are elements in S bigger than $y - \epsilon$ and $y + \epsilon \notin S$. Thus $(y-\epsilon)^2 < a < (y+\epsilon)^2$. Subtracting gives

$$(y-\epsilon)^2 - (y+\epsilon)^2 < y^2 - a < (y+\epsilon)^2 - (y-\epsilon)^2.$$

Therefore

$$|y^2 - a| < (y+\epsilon)^2 - (y-\epsilon)^2 = 4y\epsilon.$$

But this has to hold for any ϵ such that $0 < \epsilon < y$. This can only happen if $|y^2 - a| = 0$. This is what we wanted to show. □

Notation

Notation

If $a > 0$, we let $\sqrt{a}$ be the unique positive square root of a. Of course $-\sqrt{a}$ is also a square root. If $b^2 = a$, then either $b = \sqrt{a}$ or $b < 0$. But then $-b = \sqrt{a}$. Hence $\pm\sqrt{a}$ are the only square roots of a. Naturally, we let $\sqrt{0} = 0$.

Theorem

If $x \in \mathbf{R}$ and $\epsilon > 0$, then there is an irrational number s such that $|s - x| < \epsilon$.

Proof.

It suffices to see that given $a < b$, there is an irrational number s such that $a < s < b$. I claim it suffice to see that there is some irrational number t. Then we've already seen that there is a rational number r such that $|r - (x - t)| < \epsilon$. But $|r - (x - t)| = |(r + t) - x|$ and $r + t$ is irrational.
To show that there is an irrational number, the text gives a clever arguement using infinite decimal expansions. We can also show directly that $\sqrt{2}$ is irrational. We do this next. □

Proposition

$\sqrt{2}$ is irrational.

Proof.

Suppose to the contrary that $\sqrt{2} = \frac{p}{q}$ with $p, q \in \mathbf{N}$. We can assume that p and q are not both even. Then $2q^2 = p^2$. This forces p to be even. Say $p = 2k$. Then $q^2 = 2k^2$. This forces q to be even. This contradicts our assumptions on p and q. Hence $\sqrt{2}$ is irrational. □

Enough

1. That is enough for today.