

# Math 63: Winter 2021

## Lecture 3

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# Getting Started

- 1 We should be recording.
- 2 Our First Homework assignment is due Friday at 3pm via gradescope.
- 3 Time for some questions!

## Definition

We say that a field  $\mathbf{F}$  is **ordered** if there is a subset  $P \subset \mathbf{F} \setminus \{0\}$  such that

- 1  $\mathbf{F}$  is the disjoint union of  $P$ ,  $\{0\}$ , and  $-P$ .
- 2 If  $a, b \in P$ , then  $a + b \in P$  and  $ab \in P$ .

We say that  $x > 0$  if  $x \in P$ , and that  $x < y$  if  $y - x \in P$ . We call the pair  $(\mathbf{F}, P)$ , or sometimes  $(F, <)$  an **ordered field**.

## Example

Let  $P = \left\{ \frac{a}{b} \in \mathbf{Q} : a, b \in \mathbf{N} \right\}$ . Then  $(\mathbf{Q}, P)$  is an ordered field that we've held dear to our hearts since grade school.

# Not a Rational World

## Remark

We know, or at least we've been told, that rational numbers do not suffice for the sort of “real world” applications we love. Assuming, as we'll show later, that  $\sqrt{2}$  is not rational, the distance from  $(1, 0)$  to  $(0, 1)$  in the plane is not rational. The area of a circle of radius 1 is not rational. The function  $f(x) = x^2 - 2$  does not cross the  $x$ -axis in a rational world.

Since back in the day, we have been told that the rational numbers are actually a subfield of a field,  $\mathbf{R}$ , called the field of **real numbers** that contains all the numbers we want such as  $\sqrt{2}$ ,  $\pi$ ,  $e$ , as well as things we haven't even thought of yet. So what makes  $\mathbf{R}$  special?

# Least Upper Bounds

## Definition

Let  $S \subset \mathbf{F}$  be a subset of an ordered field. We say that  $b$  is an **upper bound** for  $S$  if  $s \leq b$  for all  $s \in S$ . We say that  $u$  is a **least upper bound** for  $S$  if  $u$  is an upper bound for  $S$  such that if  $t$  is any other upper bound for  $S$  then  $u \leq t$ . If  $u$  exists, then we write  $u = \text{l. u. b.}(S)$ .

## Example

Let  $S = \{1 - \frac{1}{n} : n \in \mathbf{N}\}$ . Then  $1 = \text{l. u. b.}(S)$ . Note that  $1 \notin S$ .

## Example

Let  $\mathbf{F} = \mathbf{Q}$  and  $S = \{r \in \mathbf{Q} : r^2 < 2\}$ . Then  $b = 2$  is an upper bound for  $S$  in  $\mathbf{Q}$ . But assuming  $\sqrt{2}$  is irrational and that we can find rational numbers as close as we please to  $\sqrt{2}$ , then  $S$  has no least upper bound in  $\mathbf{Q}$ . Of course, in  $\mathbf{R}$ ,  $\text{l. u. b.}(S) = \sqrt{2}$ .

# Complete Ordered Fields

## Definition

An ordered field  $\mathbf{F}$  is called **complete** if every **nonempty** set that is bounded above has a least upper bound.

## Definition

We assume that there is a complete ordered field,  $\mathbf{R}$ , called the **real numbers**, containing  $\mathbf{Q}$  as a subfield.

## Remark (Aside)

Actually, we don't have to assume  $\mathbf{Q} \subset \mathbf{R}$ . We call a subset  $A \subset \mathbf{R}$  inductive if  $1 \in A$  and  $n \in A$  implies  $n + 1 \in A$ . Since  $\mathbf{N} = \bigcap \{ A : A \text{ is inductive} \}$  is inductive, there is a smallest inductive set, namely  $\mathbf{N}$ , in  $\mathbf{R}$ . We let  $\mathbf{Z} = -\mathbf{N} \cup \{0\} \cup \mathbf{N}$  and define  $\mathbf{Q}$  as before.

# Some Real Properties

## Lemma

*If  $x \in \mathbf{R}$ , then there is a  $n \in \mathbf{N}$  such that  $n > x$ .*

## Proof.

Suppose that result is false. Then  $x$  is an upper bound for the set  $\mathbf{N}$ . Let  $a = \text{l. u. b.}(\mathbf{N})$ . Then  $a - 1$  is not an upper bound for  $\mathbf{N}$ . Why? Thus there is a  $n \in \mathbf{N}$  such that  $a - 1 < n$ . But then  $a < n + 1$ . Since  $n + 1 \in \mathbf{N}$ ,  $a$  is not an upper bound for  $\mathbf{N}$ . This is a contradiction.  $\square$

## Lemma

*Suppose  $\epsilon > 0$  in  $\mathbf{R}$ . Then there is a  $n \in \mathbf{N}$  such that  $\frac{1}{n} < \epsilon$ .*

## Proof.

Use the first lemma to find  $n \in \mathbf{N}$  such that  $\frac{1}{\epsilon} < n$ . Since  $0 < a < b$  implies  $\frac{1}{b} < \frac{1}{a}$  (HW), we have  $\frac{1}{n} < \epsilon$ .  $\square$

# Getting There

## Lemma

*If  $x \in \mathbf{R}$  then there is a  $n \in \mathbf{Z}$  such that  $n \leq x < n + 1$ .*

## Proof.

Let  $N \in \mathbf{N}$  be such that  $|x| \leq N$ . Then  $-N \leq x \leq N$ . Since  $F = \{-N, -N + 1, \dots, N - 1, N\}$  is finite, we can let  $n$  be the largest  $n \in F$  such that  $n \leq x$ . □

## Lemma

*Let  $x \in \mathbf{R}$  and  $N \in \mathbf{N}$ . Then there is a  $n \in \mathbf{Z}$  such that  $\frac{n}{N} \leq x < \frac{n+1}{N}$ .*

## Proof.

Use the previous result to find  $n \in \mathbf{Z}$  such that  $n \leq Nx < n + 1$ . □



# The Rationals are Dense

## Proposition

Suppose  $x \in \mathbf{R}$  and  $\epsilon > 0$ . Then there is a  $r \in \mathbf{Q}$  such that  $|x - r| < \epsilon$ . (Later, we will say that  $\mathbf{Q}$  is dense in  $\mathbf{R}$ .)

## Proof.

We can find  $N \in \mathbf{N}$  such that  $\frac{1}{N} < \epsilon$ . Then we can find  $n \in \mathbf{Z}$  such that  $\frac{n}{N} \leq x < \frac{n+1}{N}$ . Then  $0 \leq x - \frac{n}{N} < \frac{1}{N} < \epsilon$ . Hence  $|x - \frac{n}{N}| < \epsilon$ . □

# Break Time

- 1 Well, that was fun.
- 2 Time for a break.

# Decimals

## Definition

Let  $D = \{0, 1, 2, \dots, 9\}$ . If  $\{a_1, \dots, a_n\} \subset D$  and  $a_0 \in \mathbf{Z}$ , then we define

$$a_0.a_1a_2 \cdots a_n = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}. \quad (\ddagger)$$

We call  $(\ddagger)$  a **finite decimal expansion**.

## Lemma (Rounding Off)

Let  $a_0.a_1 \cdots a_n$  be a finite decimal expansion. If  $1 \leq m < n$ , then

$$a_0.a_1 \cdots a_m \leq a_0.a_1 \cdots a_n < a_0.a_1 \cdots a_m + \frac{1}{10^m}.$$

## Remark

For example,  $1.23 \leq 1.2345 < 1.23 + 0.01$ . (Here,  $m = 2$ .)

## Proof of the Lemma.

We have

$$\begin{aligned} a_0 \cdot a_1 \cdots a_m &\leq a_0 \cdot a_1 \cdots a_n \\ &= a_0 \cdot a_1 \cdots a_m + a_{m+1} 10^{-(m+1)} + \cdots + a_n 10^{-n} \\ &\leq a_0 \cdot a_1 \cdots a_m + 9 \cdot 10^{-(m+1)} + \cdots + 9 \cdot 10^{-n} \end{aligned}$$

which, after adding  $10^{-n} > 0$  to the right-hand side, is

$$\leq a_0 \cdot a_1 \cdots a_m + 10^{-m}.$$



# Infinite Decimals

Given  $a_0 \in \mathbf{Z}$  and a sequence  $(a_n) \subset \{0, 1, 2, \dots, 9\}$ , consider the set

$$S = \{ a_0.a_1 \cdots a_n : n \in \mathbf{N} \}.$$

This set is nonempty bounded above by  $a_0.a_1 + 0.1$ . Hence we can define

$$a_0.a_1a_2 \cdots = \text{l. u. b.} \{ a_0.a_1 \cdots a_n : n \in \mathbf{N} \}.$$

We call  $a_0.a_1a_2 \cdots$  an **infinite decimal expansion**.

## Theorem

*If  $x \in \mathbf{R}$ , then for some  $a_0 \in \mathbf{Z}$  and sequence  $(a_n) \subset \{0, 1, \dots, 9\}$  we have  $x = a_0.a_1a_2 \cdots$ . We say that  $a_0.a_1a_2 \cdots$  is a decimal expansion for  $x$ .*

## Sketch of the Proof.

Given  $m \in \mathbf{N}$ , there is a  $n \in \mathbf{Z}$  such that  $\frac{n}{10^m} \leq x \leq \frac{n+1}{10^m}$ . Thus  $a_0.a_1 \cdots a_m \leq x \leq a_0.a_1 \cdots a_m + 10^{-m}$  for appropriate  $a_k$ . With more work than I want to put in here, we can see that the  $a_k$  are uniquely determined by  $x$ . Then, replacing  $m$  by  $m+1$ , we can find  $a_{m+1}$  such that

$$a_0.a_1 \cdots a_m a_{m+1} \leq x \leq a_0.a_1 \cdots a_m a_{m+1} + 10^{-(m+1)}.$$

Continuing inductively, we get a sequence  $(a_k)$  such that

$$x = \text{l. u. b.} \{ a_0.a_1 \cdots a_m : m \in \mathbf{N} \} = a_0.a_1 a_2 \cdots . \quad \square$$

## Remark

Our theorem implies that every real number has an infinite decimal expansion, but for example,  $1.2499999 \dots = 1.2500000 \dots = 1.25$ . But this is all that can go wrong.

- 1 Let's take a break.



# Square Roots

## Theorem

*If  $a \in \mathbf{R}$  and  $a > 0$ , then there is a unique  $y > 0$  such that  $y^2 = a$ . Colloquially, every positive real number has a unique positive square root.*

## Proof.

If  $0 < x < y$ , then  $x^2 < y^2$ . Hence if  $a$  a positive square root, then it is unique. Thus we just have to show existence.

Let  $S = \{x \geq 0 : x^2 \leq a\}$ . Since  $0 \in S$ ,  $S$  is not empty. Let  $b = \max\{a, 1\}$ . Then  $b^2 = b \cdot b \geq b \cdot 1 = b \geq a$ . Hence  $b$  is an upper bound for  $S$ . Let  $y = \text{l. u. b.}(S)$ . We want to show that  $y^2 = a$ .

## Proof Continued.

If  $b = \min\{a, 1\}$ , then  $b^2 = b \cdot b \leq b \cdot 1 = b \leq a$ . Hence  $b \in S$ .

Thus  $y > 0$ .

Now suppose  $0 < \epsilon < y$ . Then  $0 < y - \epsilon < y < y + \epsilon$  and  $(y - \epsilon)^2 < y^2 < (y + \epsilon)^2$ . Since  $y = \text{l. u. b.}(S)$ , there are elements in  $S$  bigger than  $y - \epsilon$  and  $y + \epsilon \notin S$ . Thus

$(y - \epsilon)^2 < a < (y + \epsilon)^2$ . Subtracting gives

$$(y - \epsilon)^2 - (y + \epsilon)^2 < y^2 - a < (y + \epsilon)^2 - (y - \epsilon)^2.$$

Therefore

$$|y^2 - a| < (y + \epsilon)^2 - (y - \epsilon)^2 = 4y\epsilon.$$

But this has to hold for any  $\epsilon$  such that  $0 < \epsilon < y$ . This can only happen if  $|y^2 - a| = 0$ . This is what we wanted to show.  $\square$

## Notation

If  $a > 0$ , we let  $\sqrt{a}$  be the unique positive square root of  $a$ . Of course  $-\sqrt{a}$  is also a square root. If  $b^2 = a$ , then either  $b = \sqrt{a}$  or  $b < 0$ . But then  $-b = \sqrt{a}$ . Hence  $\pm\sqrt{a}$  are the only square roots of  $a$ . Naturally, we let  $\sqrt{0} = 0$ .

# Irrationals

Of course, a real number that is not rational is called an irrational number.

## Theorem

*If  $x \in \mathbf{R}$  and  $\epsilon > 0$ , then there is an irrational number  $s$  such that  $|s - x| < \epsilon$ .*

## Proof.

I claim it suffices to see that there is some irrational number  $t$ . Then we've already seen that there is a rational number  $r$  such that  $|r - (x - t)| < \epsilon$ . But  $|r - (x - t)| = |(r + t) - x|$  and  $r + t$  is irrational.

So we just have to exhibit an irrational number. We do this next. □

### Proposition

$\sqrt{2}$  is irrational.

### Proof.

Suppose to the contrary that  $\sqrt{2} = \frac{p}{q}$  with  $p, q \in \mathbf{N}$ . We can assume that  $p$  and  $q$  are not both even. Then  $2q^2 = p^2$ . This forces  $p$  to be even. Say  $p = 2k$ . Then  $q^2 = 2k^2$ . This forces  $q$  to be even. This contradicts our assumptions on  $p$  and  $q$ . Hence  $\sqrt{2}$  is irrational.  $\square$

# Enough

- 1 That is enough for today.