Math 63: Winter 2021
Lecture 4

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Getting Started

1. We should be recording.
2. If you have not turned in the first assignment, it is late!
3. There is no class Monday (January 18th) due to the MLK holiday.
4. Time for some questions!

Definition

If E is a set, then a metric on E is a function $d : E \times E \to [0, \infty)$ such that

1. $d(p, q) = 0$ if and only if $p = q$.
2. For all $p, q \in E$, $d(p, q) = d(q, p)$.
3. For all $p, q, r \in E$, we have $d(p, r) \leq d(p, q) + d(q, r)$.

The pair (E, d) is called a metric space.

Remark

If (E, d) is a metric space, the elements of E are referred to as "points" and $d(p, q)$ as the "distance from p to q. Often, we get lazy and just say that "E is a metric space" and assume that we know which metric we are talking about.

Key Examples

Examples (Our Favorite Examples)

1. Let $E = \mathbf{R}$ and $d(x, y) = |x - y|$. Then properties 1 and 2 are immediate while $d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$ gives us the triangle inequality.
2. More generally, we let *n-dimensional Euclidean space* $\mathbf{R}^n = \{\, x = (x_1, \dots, x_n) : x_k \in \mathbf{R} \,\}$ of n-tuples of real numbers with the metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Usually we will write E^n in place of $(\mathbf{R}^n, d)$. Note that E^1 is simply $(\mathbf{R}, d)$ as in item 1 above.

One Small Problem

Remark

This is all well and good, but is d a metric on $\mathbf{R}^n$ if $n \geq 2$? To verify this, it will be helpful to bring in some notation from our multivariable calculus days. If $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, then let

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} = d(x, 0).$$

If we also have $y = (y_1, \dots, y_n) \in \mathbf{R}^n$ then we let

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

Cauchy-Schwarz Inequality

Theorem ((Cauchy-)Schwarz Inequality)

If $x = (x_1, \ldots, y_n)$ *and* $y = (y_1, \ldots, y_n)$ *are in* $\mathbf{R}^n$*, then*

$$|x \cdot y| \leq \|x\|_2 \|y\|_2.$$

Proof.

The result is clear if either $\|x\|_2 = 0$ or $\|y\|_2 = 0$. So we can assume both are non-zero. For any $a, b \in \mathbf{R}$, we have

$$\begin{aligned} 0 \leq \|ax - by\|^2 &= (ax_1 - by_1)^2 + \cdots + (ax_n - by_n)^2 \\ &= a^2x_1^2 - 2abx_1y_1 + b^2y_1^2 + \cdots a^2x_n^2 - 2abx_ny_n + b^2y_n^2 \\ &= a^2\|x\|_2^2 - 2ab(x \cdot y) + b^2\|y\|_2^2. \end{aligned}$$

Proof

Proof Continued.

Hence for all $a, b \in \mathbf{R}$ we have

$$2ab(x \cdot y) \leq a^2\|x\|_2^2 + b^2\|y\|_2^2.$$

Thus if we let $a = \|y\|_2$ and $b = \pm\|x\|_2$, then

$$\pm\, 2\|x\|_2\|y\|_2(x \cdot y) \leq 2\|x\|_2^2\|y\|_2^2$$

Since we are assuming $\|x\|_2\|y\|_2 \neq 0$ we can divide both sides by $2\|x\|_2\|y\|_2$ to get

$$\pm\,(x \cdot y) \leq \|x\|_2\|y\|_2.$$

This completes the proof. □

A Corollary

Corollary

Suppose $a = (a_1, \ldots, a_n)$ *and* $b = (b_1, \ldots, b_n)$ *are in* $\mathbf{R}^n$*. Then*

$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \|a\|_2 + \|b\|_2.$$

Proof.

Since $(a_k + b_k)^2 = a_k^2 + 2a_k b_k + b_k^2$, we have

$$\begin{aligned}(LHS)^2 &= \|a\|_2^2 + 2(a \cdot b) + \|b\|_2^2 \\ &\leq \|a\|_2^2 + 2\|a\|_2\|b\|_2 + \|b\|_2^2 = \big(\|a\|_2 + \|b\|_2\big)^2.\end{aligned}$$

Now we take the square root of both sides and observe that $0 \leq x \leq y$ implies $\sqrt{x} \leq \sqrt{y}$. Why? □

Euclidean Space

Theorem

The function $d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ *is a metric on* $\mathbf{R}^n$. *Hence* E^n *is indeed a metric space.*

Proof.

The only nontrivial bit is the triangle inequality. But if $x, y, z \in \mathbf{R}^n$, then if $a_k = (x_k - y_k)$ and $b_k = (y_k - z_k)$,

$$\begin{aligned} d(x,z) &= \sqrt{(x_1 - z_1)^2 + \cdots + (x_n - z_n)^2} \\ &= \sqrt{((x_1 - y_1) + (y_1 - z_1))^2 + \cdots + ((x_n - y_n) + (y_n - z_n))^2} \\ &= \sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \\ &\leq \|a\|_2 + \|b\|_2 = d(x,y) + d(y,z). \end{aligned}$$

□

Break Time

Time for a break and some questions.

Other Examples

Example (Subpaces)

Let (E, d) be a metric space and $F \subset E$ any subset. Then we can make F into a metric space by defining $d_F(x, y) = d(x, y)$ for all $x, y \in F$. Equipped with this metric, we all F a subspace of E. For example, consider $[0, 1] \subset \mathbf{R}$. Then $[0, 1]$ is a metric space with $d(x, y) = |x - y|$. Of course, we can take any subset of Euclidean space E^n and view it as a subspace as well. For example $B^n = \{ x \in E^n : d(x, 0) \leq 1 \}$ is a metric space (with the Euclidean metric). Of course, here 0 is short hand for the n-tuple $(0, 0, \ldots, 0)$. Later, we will call B^n the "closed unit ball in E^n".

Lots of Metrics Out There

Example (The Discrete Metric)

Let E be any set. Then define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \text{ and} \\ 1 & \text{if } x \neq y. \end{cases}$$

I leave it to you to check that d is a metric on E called the discrete metric. In Math 63, we are mostly interested in E^n and its subspaces, but like the field $\mathbb{F}_4$, it is good to know that the discrete metric is out there so that proper care is taken in proofs about general metric spaces.

Low Hanging Friut

Proposition

If $x_1, \ldots, x_n$ *are points in a metric space* (E, d)*, then*

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Proof.

Since the result is trivially true if $n = 1$, we can proceed by induction. Assume the result holds for n points with $n \geq 1$. Consider $x_1, \ldots, x_n, x_{n+1}$ Then by the usual triangle inequality,

$$d(x_1, x_{n+1}) \leq d(x_1, x_n) + d(x_n, x_{n+1})$$

which, by the inductive hypotheses, is

$$\leq d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

This completes the proof. □

Reverse Triangle Inequality

Proposition (Reverse Triangle Inequality)

If (E, d) is a metric space , then for all $x, y, z \in E$ we have

$$\big|d(x,z) - d(y,z)\big| \leq d(x,y)$$

Proof.

We have $d(x,z) \leq d(x,y) + d(y,z)$. Hence

$$d(x,z) - d(y,z) \leq d(x,y).$$

But symmetry

$$d(y,z) - d(x,z) \leq d(y,x) = d(x,y).$$

The result follows. □

Break Time

Time for another break and a few questions.

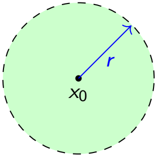
Definition

Suppose (E, d) is a metric space. If $x_0 \in E$ and $r > 0$, then the open ball of radius r centered at x_0 is

$$B_r(x_0) = \{\, y \in E : d(y, x_0) < r \,\}.$$

The closed ball of radius r centered at x_0 is

$$\{\, y \in E : d(y, x_0) \leq r \,\}.$$



In $\mathbf{R}$ with the usual metric, the open ball is just the open interval $(x_0 - r, x_0 + r)$. In E^2, open balls are really disks as drawn at left. Note that the boundary is not included. In E^3, open balls are really balls (or solid spheres). In E^4, your guess is as good as mine. What happens with the

Topology

Definition

A subset U of a metric space E is open if given $x \in U$ there is a $r > 0$ such that $B_r(x) \subset U$.

Theorem

Suppose that E is a metric space.

1. *Both E and $\emptyset$ are open in E.*
2. *If $U_1, \dots, U_n$ are open in E, then so is $\bigcap_{i=1}^n U_i$.*
3. *If $\{ U_i \}_{i \in I}$ is any collection of open sets in E, then $\bigcup_{i \in I} U_i$ is also open.*

Remark

Colloquially, item 2 is "the collection of open sets is closed under finite intersection" while item 3 says "the collection of open sets is closed under arbitrary unions". Note that to establish item 2, it is enough to see that the intersection of two open sets is open. Why?

Proof

Proof.

(1) Clearly, E is open since $B_1(x) \subset E$ for all $x \in E$. The proof that $\emptyset$ is open is also easy—if troubling. There are no points in $\emptyset$, so the condition is vacuously satisfied.

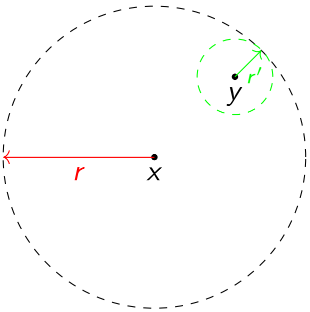
(2) Suppose that U_k is open for $k = 1, 2, \dots, n$ and that $x \in \bigcap_k U_k$. Then there is a $r_k > 0$ such that $B_{r_k}(x) \subset U_k$. Let $r = \min\{ r_1, \dots, r_n \}$. Then $r > 0$ (since the set if finite!). Then for each $1 \leq k \leq n$, $x \in B_r(x) \subset B_{r_k}(x) \subset U_k$. Hence $B_r(x) \subset \bigcap_k U_k$. This shows that the intersection is open.

(3) This is fairly straightforward. If $x \in \bigcup U_i$, then $x \in U_{i_0}$ for some $i_0 \in I$. Since U_{i_0} is open, there is a $r > 0$ such that $B_r(x) \subset U_{i_0} \subset \bigcup_{i \in I} U_i$. □

Proof by Terminology

Proposition (Yes, this needs a proof!)

In any metric space E, an open ball is an open set.



Proof.

Suppose that $x \in E$ and $r > 0$. We want to show $B_r(X)$ is open. Let $y \in B_r(x)$. Then $r' = r - d(x, y) > 0$. Now it will suffice to see that $B_{r'}(y) \subset B_r(x)$. So suppose that $z \in B_{r'}(y)$. Then

$$d(x, z) \le d(x, y) + d(y, z) < d(x, y) + r' = r. \qquad \square$$

Enough

1. That is enough for today.