Math 63: Winter 2021
Lecture 5

Dana P. Williams

Dartmouth College

Wednesday, January 20, 2021

Getting Started

1. We should be recording.
2. We are meeting in our x-hour tomorrow from 1:40 to 2:20 to make up for Monday's missed class.
3. Time for some questions!

Topology

Definition

A subset U of a metric space E is open if given $x \in U$ there is a $r > 0$ such that $B_r(x) \subset U$.

Theorem

Suppose that E is a metric space.

1. *Both E and $\emptyset$ are open in E.*
2. *If $U_1, \dots, U_n$ are open in E, then so is $\bigcap_{i=1}^n U_i$.*
3. *If $\{ U_i \}_{i \in I}$ is any collection of open sets in E, then $\bigcup_{i \in I} U_i$ is also open.*

Remark

Colloquially, item 2 is "the collection of open sets is closed under finite intersection" while item 3 says "the collection of open sets is closed under arbitrary unions".

Closed Sets

Definition

A subset F in a metric space is closed if its complement is open.

Theorem

Suppose that E is a metric space.

1. *Both E and $\emptyset$ are closed.*
2. *If $F_1, \dots, F_n$ are closed, then so is $\bigcup_{k=1}^{n} F_k$.*
3. *If F_i is closed for all $i \in I$, then so is $\bigcap_{i \in I} F_i$.*

Proof.

These facts are easily derived for the corresponding facts for open sets and taking complements. For example, item 1 follows since $\mathscr{C}E = \emptyset$ and $\mathscr{C}\emptyset = E$.

Proof

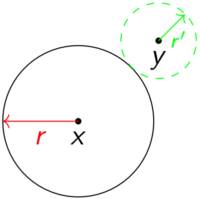
Proof Continued.

(2) If F_k is closed for $1 \leq k \leq n$, then $\mathscr{C} F_k$ is open as in $\mathscr{C}\big(\bigcup_{k=1}^n F_k\big) = \bigcap_{k=1}^n \mathscr{C} F_k$ which is open by our previous result.
(3) The proof is similar. We have $\mathscr{C}\big(\bigcap_{i\in I} F_i\big) = \bigcup_{i \in I} \mathscr{C} F_i$. $\square$

Closed Balls

Proposition

In any metric space E, every closed ball is a closed set.



Proof.

Let F be the closed ball $\{\, y \in E : d(y,x) \leq r \,\}$. We need to see that $\mathscr{C}(F)$ is open. Let $y \in \mathscr{C}(F)$. Then $r' = d(x,y) - r > 0$. Now it will suffice to see that $B_{r'}(y) \subset \mathscr{C}(F)$.
So suppose that $z \in B_{r'}(y)$. Then

$$d(x,z) \geq |d(x,y) - d(y,z)| \geq d(x,y) - d(y,z)$$

$$> d(x,y) - r' = r.$$

Hence $z \in \mathscr{C}(F)$. □

Break Time

Time for a short break and some questions.

Low Hanging Fruit

Lemma

If E is a metric space, then every finite subset $F \subset E$ is closed in E. In particular, points are closed sets in E.

Remark

The last statement means that if $x \in E$, then $\{x\}$ is a closed subset of E. But too much pedantry is not a good thing.

Proof.

Suppose $x \in E$. Let $y \in \mathscr{C}\{x\} = E \setminus \{x\}$. Then $r = d(x, y) > 0$ and $B_r(y)$ does not contain x. That is $B_r(y) \subset \mathscr{C}\{x\}$. Thus $\mathscr{C}\{x\}$ is open and $\{x\}$ is closed.

But then finite subsets of E since finite unions of closed sets are closed. □

Spheres

Example

If E is a metric space with $x \in E$ and $r > 0$, then $F = \{ y \in E : d(y, x) = r \}$ might be called the "sphere of radius r centered at x". I claim that F is closed in E.

Solution

Since $B_r(x)$ is open, its complement is closed. But $F' = \{ y \in E : d(y, x) \leq r \}$ is closed and $F = F' \cap \mathscr{C} B_r(x)$. Hence F is closed as claimed.

Most Sets are Neither Closed or Open

Example

Let $E = \mathbf{R}$ with the usual metric. Then open intervals are open sets while closed intervals are closed sets. However "half-open" intervals like $(a, b]$ (with $a < b$) are neither closed nor open.

Solution

If $b > a$ and $2r = b - a$, then your can verify that $(a, b) = B_r(\frac{a+b}{2})$. Thus (a, b) is an open ball (hence open) and $[a, b]$ a closed ball (hence closed). But every open ball centered at b contains points strictly bigger that b, so $(a, b]$ is not open. On the other hand, every open ball centered at the point a contains points in $(a, b]$. Hence the complement of $(a, b]$ is not open so $(a, b]$ is not closed.

Rectangles

Proposition

Let $1 \leq k \leq n$ and $a \in \mathbf{R}$. Then

$$U_k = \{\, (x_1, \dots, x_n) \in E^n : x_k > a \,\} \quad \text{and} \quad V_k = \{\, (x_1, \dots, x_n) \in E^n : x_k < a \,\}$$

are both open in E^n.

Proof.

Suppose that $x = (x_1, \dots, x_n) \in U_k$. Then $r = x_k - a > 0$. Suppose that $y = (y_1, \dots, y_n) \in B_r(x)$. Then

$$|x_k - y_k| \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = d(x, y) < x_k - a.$$

Then $y_k = x_k - (x_k - y_k) \geq x_k - |x_k - y_k| > x_k - (x_k - a) = a$. Therefore $y \in U_k$. This shows that U_k is open and the proof for V_k is similar. $\square$

Rectangles

Corollary

For all $1 \leq k \leq n$ *and* $a \in \mathbf{R}$ *the sets*

$$F_k = \{\, (x_1, \dots, x_n) \in E^n : x_k \geq a \,\} \quad \textit{and} \qquad F'_k = \{\, (x_1, \dots, x_n) \in E^n : x_k \leq a \,\}$$

are closed.

Proof.

Take complements. $\square$

Rectangles

Corollary

The set

$$\{\, x \in E^n : a_k < x_k < b_k \text{ for all } 1 \le k \le n \,\} \qquad (1)$$

is open in E^n. *We call* (1) *an* *open interval* *in* E^n. *Similarly, the set*

$$\{\, x \in E^n : a_k \le x_k \le b_k \text{ for all } 1 \le k \le n \,\} \qquad (2)$$

is closed in E^n. *We call* (2) *a* *closed interval* *in* E^n. return

Proof.

Just take intersections. □

Remark

Draw some pictures in E^2 and even E^3 if you are better at drawing than me.

Break Time

Time for a short break and some questions.

Bounded Sets

Definition

A subset S of a metric space E is called bounded if S is contained in some ball.

Example

Every open or closed interval in E^n is bounded.

Solution

It suffices to consider a closed interval such as (2) above*. (Why?) Let* $a = (a_1, \dots, a_n)$*,* $b = (b_1, \dots, b_n)$*, and* $r > d(a, b)$*. Then the closed interval is contained in* $B_r(a)$*.*

Low Hanging Fruit

Remark

Suppose that E is a metric space and $x_0 \in E$. If $S \subset E$ is bounded, then by definition, there is a $r > 0$ and $x \in E$ such that $S \subset B_r(x)$. Let $r_0 = d(x, x_0) + r$. Then it is not hard to check that $S \subset B_{r_0}(x_0)$. Now you can prove that the finite union of bounded subsets of E is bounded. (Explain!)

Remark

A subset of $\mathbf{R}$ is bounded if and only if it is bounded above and bounded below.

Closed Bounded Sets

Theorem

Let F be a closed nonempty subset of a metric space E. If F is bounded above, then F has a largest element. Similarly, if F is bounded below, then F has a least element.

Proof.

Suppose that F is bounded above. Then since F is nonempty, we can let $u = \text{l.u.b.}(F)$. If $u \in F$, then we are done. So assume to the contrary that $a \notin F$. Since F is closed, $\mathscr{C}F$ is open and $a \in \mathscr{C}F$. Hence there is a $r > 0$ such that $(u - r, u + r) \subset \mathscr{C}F$. But then $x \in F$ implies $x \leq u - r$ and $u - r$ is an upper bound for F. This contradicts the definition of u.

We get the result for least elements by replacing F by $-F$. $\square$

Break Time

Time for a break and a few questions

Definition

Let (x_n) be a sequence in a metric space (E, d). We say that (x_n) converges to $x \in E$ if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon$. We say that x is a limit of (x_n).

Remark

Of course, the $N \in \mathbf{N}$ above will depend on $\epsilon > 0$. Some folks get pedantic and write $N(\epsilon)$ to indicate this. We don't really care what N is—just that it exists. If we find an N that works for a give ϵ, then any larger N will work just as well.

Example

Example

Consider the sequence $(x_n) \subset \mathbf{R}$ with $x_n = \frac{1}{n}$. Then given $\epsilon > 0$, we know there is a $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$. Thus $n \geq N$ implies $|\frac{1}{n} - 0| = d(x_n, 0) < \epsilon$. Therefore $(\frac{1}{n})$ converges to 0 in $\mathbf{R}$. However, if we replace $\mathbf{R}$ with $E = (0, 1)$, then $(\frac{1}{n})$ is a sequence in E, but it does not converge to 0 simply because $0 \notin E$!

Remark

In the standard picture—for example, in the pictures we draw in our calculus classes—we think of the points in a convergent sequence (x_n) getting ever closer to the limit, say x. However, this is not strictly true. Consider the sequence $1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots, x_n, \dots$ where

$$x_n = \begin{cases} \frac{1}{k} & \text{if } n = 2k - 1, \text{ and} \\ 0 & \text{if } n = 2k. \end{cases}$$

We could also take the constant sequence $x, x, x, \dots$ which is easily seen to converge to x.

Unique Limits

Proposition

A sequence (x_n) in a metric space E has at most one limit.

Proof.

Suppose that (x_n) converges to both x and y. If $x \neq y$, then we can let $\epsilon = d(x, y)$. Then there is a N_1 such that $n \geq N_1$ implies $d(x_n, x) < \frac{\epsilon}{2}$. Similarly, there is a N_2 such that $n \geq N_2$ implies that $d(x_n, y) < \frac{\epsilon}{2}$. Let $n = \max\{ N_1, N_2 \}$. Then $d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = d(x, y)$. This is a contradiction, so $x = y$. $\square$

Notation

Notation

If (x_n) is a sequence in a metric space E converging to $x \in E$, then we write

$$\lim_{n\to\infty} x_n = x.$$

We may get sloppy and write $x_n \to x$ as $n \to \infty$ or just $x_n \to x$. If (x_n) does not have a limit, then we say that $\lim_{n\to\infty} x_n$ "does not exist" or that (x_n) "diverges".

Enough

1. That is enough for today.