Math 63: Winter 2021
Lecture 6

Dana P. Williams

Dartmouth College

Thursday January 21, 2021

Getting Started

1. We should be recording.
2. The second homework assignment is due tomorrow before lecture starts via `gradescope`.
3. Time for some questions!

Some Necessary Repairs

Proposition

Let $1 \leq k \leq n$ and $a \in \mathbf{R}$. Then

$$U_k = \{ (x_1, \dots, x_n) \in E^n : x_k > a \} \quad \text{and} \quad V_k = \{ (x_1, \dots, x_n) \in E^n : x_k < a \}$$

are both open in E^n.

Proof.

Suppose that $x = (x_1, \dots, x_n) \in U_k$. Then $r = x_k - a > 0$. Suppose that $y = (y_1, \dots, y_n) \in B_r(x)$. Then

$$|x_k - y_k| \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = d(x, y) < x_k - a.$$

Then $y_k = x_k - (x_k - y_k) \geq x_k - |x_k - y_k| > x_k - (x_k - a) = a$. Therefore $y \in U_k$. This shows that U_k is open and the proof for V_k is similar. $\square$

Last Time

Definition

Let (x_n) be a sequence in a metric space (E, d). We say that (x_n) converges to $x \in E$ if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon$. We say that x is a limit of (x_n).

Proposition

A sequence (x_n) in a metric space E has at most one limit.

Notation

If (x_n) is a sequence in a metric space E converging to $x \in E$, then we write

$$\lim_{n \to \infty} x_n = x.$$

We may get sloppy and write $x_n \to x$ as $n \to \infty$ or just $x_n \to x$. If (x_n) does not have a limit, then we say that $\lim_{n\to\infty} x_n$ "does not exist" or that (x_n) "diverges".

Subsequences

Definition

Let (x_n) be a sequence. Let $n_1 < n_2 < \dots$ be a strictly increasing sequence in $\mathbf{N}$. Then the sequence $(x_{n_k})_{k=1}^{\infty}$ is called a subsequence of (x_n).

Example

Let (x_n) be a sequence in $\mathbf{R}$. Then the even terms (x_{2n}) form a subsequence. Thus if $x_n = (-1)^2$, then the constant sequence $1, 1, 1, \dots$ is a subsequence of $-1, 1, -1, 1, \dots$.

Convergence of Subsequences

Proposition

Suppose that (x_n) is a sequence in a metric space E which converges to $x \in E$. Then every subsequence of (x_n) also converges to x.

Proof.

Suppose $\lim_{n\to\infty} x_n = x$ and that (x_{n_k}) is a subsequence. Let $\epsilon > 0$. Then there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon$. Notice that we always have $n_k \geq k$. (This follows from a simple induction argument.) Hence if $k \geq N$, then $n_k \geq N$ and $d(x_{n_k}, x) < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have shown that $\lim_{k\to\infty} x_{n_k} = x$. □

Bounded Sequences

Definition

We say that a sequence (x_n) in a metric space E is bounded if the set $\{ x_n : n \in \mathbf{N} \}$ is bounded.

Proposition

A convergent sequence in a metric space is bounded.

Proof.

Suppose (x_n) converges to x. Let $N \in \mathbf{N}$ be such that $n \geq N$ implies $d(x_n, x) < 1$. Let $M = \max\{ d(x_1, x), \dots, d(x_N, x), 1 \} + 1$. Then $\{ x_n : n \in \mathbf{N} \} \subset B_M(x)$ and $\{ x_n : n \in \mathbf{N} \}$ is bounded. $\square$

Closed Sets and Sequences

Theorem

Let F be a subset of a metric space E. Then F is closed if and only if whenever we have a sequence (x_n) of points in F converging to $x \in E$ it follows that $x \in F$.

Proof.

First suppose that F is closed. Let (x_n) be a sequence of points in F converging to $x \in E$. Suppose contrary to what we which to prove that $x \notin F$. Then $x \in \mathscr{C}F$. Since $\mathscr{C}F$ is open, there is a $r > 0$ such that $B_r(x) \subset \mathscr{C}F$. But there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $x_n \in B_r(x)$. But then $x_N \in \mathscr{C}F$ which is a contradiction.

Proof

Proof Continued.

Conversely, suppose that convergent sequences in F must converge to a point in F. Suppose to the contrary of what we want to show that F is not closed. Then $\mathscr{C}F$ is not open. Thus there is a $x \in \mathscr{C}F$ such that $B_r(x) \not\subset \mathscr{C}F$ for all $r > 0$. That is, $B_r(x) \cap F \neq \emptyset$ for all $r > 0$. Then we can find $x_n \in B_{\frac{1}{n}}(x) \cap F$ for all $n \in \mathbf{N}$. Now it is easy to show that (x_n) converges to x. Since (x_n) is in F and $x \notin F$, this is a contradiction. □

Break Time

Time for a short break and questions

The Real Field Again

Theorem

Suppose that (a_n) and (b_n) are sequences of real numbers such that $\lim_n a_n = a$ and $\lim_n b_n = b$. Then

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$,
2. $\lim_{n\to\infty}(a_n - b_n) = a - b$,
3. $\lim_{n\to\infty} a_n b_n = ab$, *and*
4. *provided b and all the b_n are not equal to* 0, $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof.

(1) Let $\epsilon > 0$. Let $N_1 \in \mathbf{N}$ be such that $n \geq N_1$ implies $|a_n - a| < \frac{\epsilon}{2}$ and $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $|b_n - b| < \frac{\epsilon}{2}$. Let $N = \max\{ N_1, N_2 \}$. Then if $n \geq N$ we have $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows $a_n + b_n \to a + b$.

Proof

Proof Continued.

(3) Since convergent sequences are bounded, there is a $M > 0$ such that $|a_n| \le M$ and $|b_n| \le M$ for all $n \in \mathbf{N}$. Since $\{ x \in \mathbf{R} : |x| \le M \}$ is closed, we have $|a| \le M$ and $|b| \le M$ as well. Thus if $\epsilon > 0$, there is a $N \in \mathbf{N}$ such that $n \ge N$ implies $|a_n - a| < \frac{\epsilon}{2M}$ and $|b_n - b| < \frac{\epsilon}{2M}$. Thus if $n \ge N$, we have

$$\begin{aligned} |a_n b_n - ab| &\le |a_n b_n - ab_n + ab_n - ab| \\ &\le |a_n - a||b_n| + |a||b_n - b| \\ &\le \frac{\epsilon}{2M} M + \frac{\epsilon}{2M} M = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $a_n b_n \to ab$.

Proof

Proof Continued.

(2) Of course, we could just mimic the proof of (1). However, $-b_n = (-1)b_n$, so we can use part (3) to conclude that $-b_n = (-1)b_n \to (-1)b = -b$. Now by part (1), $a_n - b_n = a_n + (-b_n) \to a + (-b) = a - b$.

(4) We'll first show that $\frac{1}{b_n} \to \frac{1}{b}$. Then we can apply part (3): $\frac{a_n}{b_n} = a_n \frac{1}{b_n} \to a\frac{1}{b} = \frac{a}{b}$.

For motivation, observe that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|}.$$

We have to be sure that the denominator doesn't get too small! We can accomplish this if we ensure that $|b - b_n| < \frac{|b|}{2}$.

Out of Time

We ran out of time. We will pick this up tomorrow.