Math 63: Winter 2021
Lecture 7

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Getting Started

1. We should be recording.
2. Time for some questions!

Picking Up From Last Time

Theorem

Suppose that (a_n) *and* (b_n) *are sequences of real numbers such that* $\lim_n a_n = a$ *and* $\lim_n b_n = b$. *Then*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$,
2. $\lim_{n\to\infty}(a_n - b_n) = a - b$,
3. $\lim_{n\to\infty} a_n b_n = ab$, *and*
4. *provided* b *and all the* b_n *are not equal to* 0, $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$.

Remark

Yesterday, we proved items 1, 2, and 3. Let's pick continue with item 4.

Proof

Proof of (4).

(4) We'll first show that $\frac{1}{b_n} \to \frac{1}{b}$. Then we can apply part (3): $\frac{a_n}{b_n} = a_n \frac{1}{b_n} \to a \frac{1}{b} = \frac{a}{b}$.

For motivation, observe that

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b_n - b|}{|b_n||b|}.$$

We have to be sure that the denominator doesn't get too small! We can accomplish this if we ensure that $|b - b_n| < \frac{|b|}{2}$.

Proof

Proof Continued.

Fix $\epsilon > 0$. Let N be such that $n \geq N$ implies $|b_n - b| < \min\{ \frac{|b|}{2}, \frac{|b|^2\epsilon}{2} \}$. Then if $n \geq N$, we have $|b_n| = |b - (b - b_n)| \geq |b| - |b - b_n| > |b| - \frac{|b|}{2} = \frac{|b|}{2}$. Then

$$\begin{aligned} \Big|\frac{a_n}{b_n} - \frac{a}{b}\Big| &= \frac{|b_n - b|}{|b_n||b|} \\ &< \frac{\frac{|b|^2\epsilon}{2}}{|b| \cdot \frac{|b|}{2}} = \epsilon. \end{aligned}$$

This completes the proof that $\frac{a_n}{b_n} \to \frac{a}{b}$ and we agreed that this suffices. □

The Reals Again: Order

Proposition

Suppose that (a_n) and (b_n) are sequences of real numbers that converge to a and b, respectively. If $a_n \leq b_n$ for all n, then $a \leq b$.

Proof.

We have $\lim_n (b_n - a_n) = b - a$. Since $\{ x \in \mathbf{R} : x \geq 0 \}$ is closed and $b_n - a_n \geq 0$ for all n, we have $b - a \geq 0$. That is, is $b \geq a$. $\square$

Monotonic Sequences

Definition

A sequence (a_n) of real numbers is called increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$. It is called decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is called monotonic if it is either and increasing sequence or a decreasing sequence.

Bounded Monotonic Sequences

Theorem

A bounded monotonic sequence of real numbers is convergent.

Proof.

Suppose that (a_n) is a bounded increasing sequence of real numbers. Since $S = \{ a_n : n \in \mathbf{N} \}$ is bounded above, we can let $u = \text{l.u.b.}(S)$. We will show that $\lim_n a_n = u$. Let $\epsilon > 0$. Then $u - \epsilon$ is not an upper bound for S. Hence there is a $N \in \mathbf{N}$ such that $a_N > u - \epsilon$. But if $n \geq N$, we have

$$u - \epsilon < a_N \leq a_n \leq u < u + \epsilon.$$

Hence $n \geq N$ implies $|a_n - u| < \epsilon$.

If (a_n) is bounded and decreasing, consider $(-a_n)$. Then $-a_n \to u = \text{l.u.b.} -a_n : n \in \mathbf{N}$. Thus $a_n \to -u$. □

A Corollary

Corollary

Suppose that $a \in \mathbf{R}$ *and* $|a| < 1$. *Then* $\lim_{n\to\infty} a^n = 0$.

Proof.

The sequence (a^n) is decreasing: $a^{n+1} = aa^n \leq (1)a^n = a^n$. Hence $\lim_n a^n = x$ for some $x \geq 0$. But then
$ax = a(\lim_n a^n) = \lim_n aa^n = \lim_n a^{n+1} = \lim_n a^n = x$. Thus
$0 = ax - x = (a-1)x$. Since $a - 1 \neq 0$, we must have $x = 0$. $\square$

Break Time

Time for a break and a few questions.

Cauchy Sequences

Definition

A sequence (x_n) in a metric space is called Cauchy if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$.

Example

Suppose that (x_n) is a convergent sequence in a metric space E. Then (x_n) is Cauchy.

Solution

Suppose $x_n \to x$. Let $\epsilon > 0$. Let $N \in \mathbf{N}$ be such that $n \geq N$ implies $d(x_n, x) < \frac{\epsilon}{2}$. Then if $n, m \geq N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (x_n) is Cauchy.

More Cauchy

Proposition

A subsequence of a Cauchy sequence is a Cauchy sequence.

Proof.

This is an exercise. □

Proposition

A Cauchy sequence is bounded.

Proof.

Suppose (x_n) is Cauchy. Let N be such that $n, m \geq N$ implies $d(x_n, x_m) < 1$. Hence $d(x_n, x_N) < 1$ for all $n \geq N$. Let

$$M = \max\{\, d(x_1, x_N), \dots, d(x_{N-1}, x_N) \,\} + 1.$$

Then for all $n \in \mathbf{N}$, $x_n \in B_M(x_N)$. Hence $\{\, x_n : n \in \mathbf{N} \,\}$ is bounded. □

Subsequences Suffice

Proposition

Suppose that (x_n) is Cauchy and that (x_n) has a convergent subsequence. Then (x_n) is convergent.

Proof.

Suppose that (x_{n_k}) converges to x. Let $\epsilon > 0$. Let N be such that $n, m \geq N$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$. Then there is a K such that $k \geq K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$. We can assume $K \geq N$ so that $x_{n_K} \geq N$. Now $n \geq N$ implies

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we've shown $x_n \to x$. □

Completeness

Definition

A metric space E is called complete if every Cauchy sequence in E is convergent.

Example

Let $E = (0, 1]$ and consider (x_n) where $x_n = \frac{1}{n}$. Then (x_n) is Cauchy in E. (Why?) But (x_n) does not converge in E. Therefore, $E = (0, 1)$ is not a complete metric space.

The Reals are Complete

Theorem

The Reals are a complete metric space.

Remark

By assumption, the Reals are a complete ordered field. But we have to prove completeness as a metric space!

Proof.

Suppose that (x_n) is Cauchy in $\mathbf{R}$. Let $S = \{ x \in \mathbf{R} : \{ x_n : x_n < x \} \text{ is finite} \}$. Since $\{ x_n : n \in \mathbf{N} \}$ is bounded, and hence bounded below, $S \neq \emptyset$. Since $\{ x_n : n \in \mathbf{N} \}$ is also bounded above, S is also bounded above. Let $u = \text{l.u.b.}(S)$. It will suffice to show that $x_n \to u$.

Proof

Proof Continued.

Let $\epsilon > 0$. Let N be such that $n, m \geq N$ implies $|x_n - x_m| < \frac{\epsilon}{2}$. Then $u - \frac{\epsilon}{2} \in S$ and there are only finitely many n such that $a_n < u - \frac{\epsilon}{2}$. Hence there is a N such that $n \geq N$ implies $a_n \geq u - \frac{\epsilon}{2}$. On the other hand, $u + \frac{\epsilon}{2} \notin S$ and there are infinitely many n such that $a_n < u + \frac{\epsilon}{2}$. Thus there is a $m \geq N$ such that $a_m \geq u - \frac{\epsilon}{2}$ and $a_m < u + \frac{\epsilon}{2}$. Then $|a_m - u| \leq \frac{\epsilon}{2}$. Thus if $n \geq N$, we have

$$|u - a_n| \leq |u - a_m| + |a_m - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus we shown $a_n \to u$ as required. □

Closed Sets

Proposition

If E is a complete metric space and if F is a closed subspace, then F is complete.

Proof.

Suppose that (x_n) is a Cauchy sequence in F. Then (x_n) is also Cauchy in E. Since E is complete, (x_n) must converge to some $x \in E$. Since F is closed and $(x_n) \subset F$, we must have $x \in F$. Hence $x_n \to x$ in F and F is complete. □

Example

The closed interval $[0, 1]$ is complete in its natural metric.

Break Time

Time for a break and some questions.

Lemma

Let (x_m) be a sequence in E^n and let $x_m = (x_m^1, \dots, x_m^n)$. Then (x_m) converges to $x = (x^1, \dots, x^n)$ if and only if $\lim_m x_m^k = x^k$ for $1 \le k \le n$.

Proof.

Suppose that $\lim_m x_m = x$ in E^n. Notice that for all k,

$$|x_m^k - x^k| \le \sqrt{(x_m^1 - x^1)^2 + \cdots + (x_m^n - x^n)^2} = d(x_m, x).$$

Then given $\epsilon > 0$ there is a N such that $m \ge N$ implies $|x_m^k - x^k| \le d(x_m, x) < \epsilon$. Thus $x_m^k \to x^k$ for all k.

Proof

Proof Continued.

Conversely, suppose $x_m^k \to x^k$ for all k. Let N_k be such that $m \geq N_k$ implies $|x_m^k - x^k| < \frac{\epsilon}{\sqrt{n}}$. Let $N = \max\{ N_1, \dots, N_n \}$. Then if $n \geq M$,

$$d(x_m, x) = \sqrt{(x_m^1 - x^1)^2 + \cdots + (x_m^n - x^n)^2}$$
$$< \sqrt{\frac{\epsilon^2}{n} + \cdots + \frac{\epsilon^2}{n}} = \epsilon.$$

This completes the proof. □

Euclidean Space is Complete

Theorem

For all $n \in \mathbf{N}$, Euclidean space, E^n, is complete.

Proof.

Let (x_m) be a Cauchy sequence in E^n. As in the lemma, let $x_m = (x_m^1, \dots, x_m^n)$. Then, also as in the proof of the lemma, for $1 \leq k \leq n$ and $m, l \in \mathbf{N}$, we have

$$|x_m^k - x_l^k| \leq d(x_m, x_l).$$

It follows that for each $1 \leq k \leq n$, the real sequence (x_m^k) is Cauchy. Since $\mathbf{R}$ is complete, (x_m^k) is convergent for each $1 \leq k \leq n$. By the lemma, (x_m) is convergent. Hence E^n is complete. $\square$

Enough

1. That is enough for today.