Math 63: Winter 2021
Lecture 7

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Getting Started

1. We should be recording.
2. Our preliminary exam will be available on `gradescope` after class on Friday, January 29th, and must be turned in by Sunday January 31st by 10pm. You will have 150 minutes to work the exam and an additional 30 minutes to scan, link, and upload your exam to `gradescope`. The Exam will cover through §III.5 in the text which I hope to finish on Monday.
3. Time for some questions!

Picking Up From Last Time

Theorem

Suppose that (a_n) and (b_n) are sequences of real numbers such that $\lim_n a_n = a$ and $\lim_n b_n = b$. Then

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$,
2. $\lim_{n\to\infty}(a_n - b_n) = a - b$,
3. $\lim_{n\to\infty} a_n b_n = ab$, *and*
4. *provided b and all the b_n are not equal to 0,* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Remark

Yesterday, we proved items 1, 2, and 3. Let's pick continue with item 4.

Proof

Proof of (4).

(4) We'll first show that $\frac{1}{b_n} \to \frac{1}{b}$. Then we can apply part (3): $\frac{a_n}{b_n} = a_n \frac{1}{b_n} \to a \frac{1}{b} = \frac{a}{b}$.

For motivation, observe that

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b_n - b|}{|b_n||b|}.$$

We have to be sure that the denominator doesn't get too small! We can accomplish this if we ensure that $|b - b_n| < \frac{|b|}{2}$.

Proof

Proof Continued.

Fix $\epsilon > 0$. Let N be such that $n \geq N$ implies $|b_n - b| < \min\{ \frac{|b|}{2}, \frac{|b|^2\epsilon}{2} \}$. Then if $n \geq N$, we have $|b_n| = |b - (b - b_n)| \geq |b| - |b - b_n| > |b| - \frac{|b|}{2} = \frac{|b|}{2}$. Then

$$\Big|\frac{1}{b_n} - \frac{1}{b}\Big| = \frac{|b_n - b|}{|b_n||b|}$$
$$< \frac{\frac{|b|^2\epsilon}{2}}{|b| \cdot \frac{|b|}{2}} = \epsilon.$$

This completes the proof that $\frac{a_n}{b_n} \to \frac{a}{b}$ and we agreed that this suffices. $\square$

The Reals Again: Order

Proposition

Suppose that (a_n) and (b_n) are sequences of real numbers that converge to a and b, respectively. If $a_n \leq b_n$ for all n, then $a \leq b$.

Proof.

We have $\lim_n(b_n - a_n) = b - a$. Since $\{ x \in \mathbf{R} : x \geq 0 \}$ is closed and $b_n - a_n \geq 0$ for all n, we have $b - a \geq 0$. That is, is $b \geq a$. $\square$

Monotonic Sequences

Definition

A sequence (a_n) of real numbers is called increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$. It is called decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is called monotonic if it is either and increasing sequence or a decreasing sequence.

Bounded Monotonic Sequences

Theorem

A bounded monotonic sequence of real numbers is convergent.

Proof.

Suppose that (a_n) is a bounded increasing sequence of real numbers. Since $S = \{ a_n : n \in \mathbf{N} \}$ is bounded above, we can let $u = \mathrm{l.u.b.}(S)$. We will show that $\lim_n a_n = u$. Let $\epsilon > 0$. Then $u - \epsilon$ is not an upper bound for S. Hence there is a $N \in \mathbf{N}$ such that $a_N > u - \epsilon$. But if $n \geq N$, we have

$$u - \epsilon < a_N \leq a_n \leq u < u + \epsilon.$$

Hence $n \geq N$ implies $|a_n - u| < \epsilon$.

If (a_n) is bounded and decreasing, consider $(-a_n)$. Then $-a_n \to u = \mathrm{l.u.b.} -a_n : n \in \mathbf{N}$. Thus $a_n \to -u$. $\square$

A Corollary

Corollary

Suppose that $a \in \mathbf{R}$ *and* $|a| < 1$. *Then* $\lim_{n\to\infty} a^n = 0$.

Proof.

The sequence (a^n) is decreasing: $a^{n+1} = aa^n \leq (1)a^n = a^n$. Hence $\lim_n a^n = x$ for some $x \geq 0$. But then
$ax = a(\lim_n a^n) = \lim_n aa^n = \lim_n a^{n+1} = \lim_n a^n = x$. Thus
$0 = ax - x = (a-1)x$. Since $a - 1 \neq 0$, we must have $x = 0$. □

Break Time

Time for a break and a few questions.

Cauchy Sequences

Definition

A sequence (x_n) in a metric space is called Cauchy if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$.

Example

Suppose that (x_n) is a convergent sequence in a metric space E. Then (x_n) is Cauchy.

Solution

Suppose $x_n \to x$. Let $\epsilon > 0$. Let $N \in \mathbf{N}$ be such that $n \geq N$ implies $d(x_n, x) < \frac{\epsilon}{2}$. Then if $n, m \geq N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (x_n) is Cauchy.

More Cauchy

Proposition

A subsequence of a Cauchy sequence is a Cauchy sequence.

Proof.

This is an exercise. □

Proposition

A Cauchy sequence is bounded.

Proof.

Suppose (x_n) is Cauchy. Let N be such that $n, m \geq N$ implies $d(x_n, x_m) < 1$. Hence $d(x_n, x_N) < 1$ for all $n \geq N$. Let

$$M = \max\{\, d(x_1, x_N), \dots, d(x_{N-1}, x_N)\,\} + 1.$$

Then for all $n \in \mathbf{N}$, $x_n \in B_M(x_N)$. Hence $\{\, x_n : n \in \mathbf{N}\,\}$ is bounded. □

Subsequences Suffice

Proposition

Suppose that (x_n) is Cauchy and that (x_n) has a convergent subsequence. Then (x_n) is convergent.

Proof.

Suppose that (x_{n_k}) converges to x. Let $\epsilon > 0$. Let N be such that $n, m \geq N$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$. Then there is a K such that $k \geq K$ implies $d(x_{n_k}, x) < \frac{\epsilon}{2}$. We can assume $K \geq N$ so that $n_K \geq N$. Now $n \geq N$ implies

$$d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we've shown $x_n \to x$. $\square$

Definition

A metric space E is called complete if every Cauchy sequence in E is convergent.

Example

Let $E = (0,1]$ and consider (x_n) where $x_n = \frac{1}{n}$. Then (x_n) is Cauchy in E. (Why?) But (x_n) does not converge in E. Therefore, $E = (0,1]$ is not a complete metric space.

Example

Let $E = \mathbf{Q}$ viewed as a subspace of $\mathbf{R}$. Let (r_n) be a sequence of rational numbers converging to $\sqrt{2}$. Then (r_n) is Cauchy in $\mathbf{Q}$, but does not converge (in $\mathbf{Q}$). Hence $\mathbf{Q}$ is not complete.

The Reals are Complete

Theorem

The Reals are a complete metric space.

Remark

By assumption, the Reals are a complete ordered field. But we have to prove completeness as a metric space!

Proof.

Suppose that (x_n) is Cauchy in $\mathbf{R}$. Let $S = \{\, x \in \mathbf{R} : \{\, n \in \mathbf{N} : x_n < x \,\} \text{ is finite} \,\}$. Since $\{\, x_n : n \in \mathbf{N} \,\}$ is bounded, and hence bounded below, $S \neq \emptyset$. Since $\{\, x_n : n \in \mathbf{N} \,\}$ is also bounded above, S is also bounded above. Let $u = \operatorname{l.u.b.}(S)$. It will suffice to show that $x_n \to u$.

Proof

Proof Continued.

Let $\epsilon > 0$. Let N be such that $n, m \geq N$ implies $|x_n - x_m| < \frac{\epsilon}{2}$. Then $u - \frac{\epsilon}{2} \in S$ and there are only finitely many n such that $a_n < u - \frac{\epsilon}{2}$. Hence there is a N such that $n \geq N$ implies $a_n \geq u - \frac{\epsilon}{2}$. On the other hand, $u + \frac{\epsilon}{2} \notin S$ and there are infinitely many n such that $a_n < u + \frac{\epsilon}{2}$. Thus there is a $m \geq N$ such that $a_m \geq u - \frac{\epsilon}{2}$ and $a_m < u + \frac{\epsilon}{2}$. Then $|a_m - u| \leq \frac{\epsilon}{2}$. Thus if $n \geq N$, we have

$$|u - a_n| \leq |u - a_m| + |a_m - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus we shown $a_n \to u$ as required. □

Closed Sets

Proposition

If E is a complete metric space and if F is a closed subspace, then F is complete.

Proof.

Suppose that (x_n) is a Cauchy sequence in F. Then (x_n) is also Cauchy in E. Since E is complete, (x_n) must converge to some $x \in E$. Since F is closed and $(x_n) \subset F$, we must have $x \in F$. Hence $x_n \to x$ in F and F is complete. $\square$

Example

The closed interval $[0, 1]$ is complete in its natural metric.

Break Time

Time for a break and some questions.

Euclidean Space

Lemma

Let (x_m) be a sequence in E^n and let $x_m = (x_m^1, \dots, x_m^n)$. Then (x_m) converges to $x = (x^1, \dots, x^n)$ if and only if $\lim_m x_m^k = x^k$ for $1 \le k \le n$.

Proof.

Suppose that $\lim_m x_m = x$ in E^n. Notice that for all k,

$$|x_m^k - x^k| \le \sqrt{(x_m^1 - x^1)^2 + \cdots + (x_m^n - x^n)^2} = d(x_m, x).$$

Then given $\epsilon > 0$ there is a N such that $m \ge N$ implies $|x_m^k - x^k| \le d(x_m, x) < \epsilon$. Thus $x_m^k \to x^k$ for all k.

Proof

Proof Continued.

Conversely, suppose $x_m^k \to x^k$ for all k. Let N_k be such that $m \geq N_k$ implies $|x_m^k - x^k| < \frac{\epsilon}{\sqrt{n}}$. Let $N = \max\{ N_1, \dots, N_n \}$. Then if $n \geq N$,

$$\begin{aligned} d(x_m, x) &= \sqrt{(x_m^1 - x^1)^2 + \cdots + (x_m^n - x^n)^2} \\ &< \sqrt{\frac{\epsilon^2}{n} + \cdots + \frac{\epsilon^2}{n}} = \epsilon. \end{aligned}$$

This completes the proof. □

Euclidean Space is Complete

Theorem

For all $n \in \mathbf{N}$, Euclidean space, E^n, is complete.

Proof.

Let (x_m) be a Cauchy sequence in E^n. As in the lemma, let $x_m = (x_m^1, \dots, x_m^n)$. Then, also as in the proof of the lemma, for $1 \leq k \leq n$ and $m, l \in \mathbf{N}$, we have

$$|x_m^k - x_l^k| \leq d(x_m, x_l).$$

It follows that for each $1 \leq k \leq n$, the real sequence (x_m^k) is Cauchy. Since $\mathbf{R}$ is complete, (x_m^k) is convergent for each $1 \leq k \leq n$. By the lemma, (x_m) is convergent. Hence E^n is complete. $\square$

Enough

1. That is enough for today.