Math 63: Winter 2021
Lecture 8

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Getting Started

1. We should be recording.
2. Our preliminary exam will be available on `gradescope` after class on Friday, January 29th, and must be turned in by Sunday January 31st by 10pm. You will have 150 minutes to work the exam and an additional 30 minutes to scan, link, and upload your exam to `gradescope`. The Exam will cover through §III.5 in the text most of which I hope to finish today.
3. I suggest blocking out a three hour window now so that you can work undisturbed. Although officially a take home exam, it is effectively an "in class" exam where you get to pick when you take it.
4. Time for some questions!

A Corollary

Corollary

Suppose that $a \in \mathbf{R}$ *and* $|a| < 1$. *Then* $\lim_{n\to\infty} a^n = 0$.

Proof.

Since $|a^n| = |a|^n$, it suffices to assume $0 \le a < 1$. (It is not hard to see that $\lim_n |a^n| = 0$ is equivalent to $\lim_n a^n = 0$.) The sequence (a^n) is decreasing: $a^{n+1} = aa^n \le (1)a^n = a^n$. Hence $\lim_n a^n = x$ for some $x \ge 0$. But then

$ax = a(\lim_n a^n) = \lim_n aa^n = \lim_n a^{n+1} = \lim_n a^n = x$. Thus

$0 = ax - x = (a-1)x$. Since $a - 1 \ne 0$, we must have $x = 0$. $\square$

Compactness

Definition

A subset K of a metric space E is said to be compact if, given any collection $\{ U_i : i \in I \}$ of open sets in E such that $K \subset \bigcup_{i \in I} U_i$, then there is a finite subset $F \subset I$ such that $K \subset \bigcup_{i \in F} U_i$.

Remark (Jargon)

We say that a collection $\{ U_i : i \in I \}$ of subsets of E is a cover of K if $K \subset \bigcup_{i \in I} U_i$. If each U_i is an open set, then we call $\{ U_i : i \in I \}$ an open cover of K. If $J \subset I$, and $\{ U_i : i \in J \}$ still covers K, then we call $\{ U_i : i \in J \}$ a subcover of $\{ U_i : i \in I \}$. Using this language, a subset is compact if every open cover of K has a finite subcover.

Examples are Hard to Come By

Remark

If $F \subset E$ is finite, say $F = \{ x_1, \dots, x_n \}$, then F is clearly compact: Suppose $F \subset \bigcup_{i \in I} U_i$. Then for each k, we must have $x_k \in U_{i_k}$ for some $i_i \in I$. But $F \subset \bigcup_{k=1}^n U_{i_k}$. It is a fact of life that producing nontrivial examples of compact subsets—even of $\mathbf{R}$—is itself nontrivial. But baby steps for now.

Example (A Noncompact Set)

Let $S = (0,1] \subset \mathbf{R}$. Then $S \subset \bigcup_{n=1}^\infty (\frac{1}{n}, 2)$. But no finite subset of $\{ (\frac{1}{n}, 2) : n \in \mathbf{N} \}$ will do. Please note that $(0,1]$ has lots of finite open covers! The issue is to show that any open cover has a finite subcover.

An Annoyance

Remark

Of course, we say that a metric space E is compact if it is a compact subset of itself. But this raises the annoying possibility that a subspace $K \subset E$ could be a compact metric space but not a compact subset of E. I want devote a few slides to dispensing with this technicality.

Lemma

Suppose that K is a subset of a metric space E. Then U is open in K if and only if there is an open set V in E such that $U = K \cap V$.

Proof.

This is almost tautologous for open balls: if $x \in K$, then

$$B_r^K(x) = \{\, y \in K : d(y, x) < r \,\} = B_r(x) \cap K.$$

Thus if $U \subset K$ is open, then for each $x \in U$ there is a $r_x > 0$ such that $B_{r_x}^K(x) \subset U$. But then

$$U = \bigcup_{x \in U} B_{r_x}^K(x) = \bigcup_{x \in U} (B_{r_x}(x) \cap K) = \Big(\bigcup_{x \in U} B_{r_x}(x)\Big) \cap K$$

and we can let $V = \bigcup_{x \in U} B_{r_x}(x)$. $\square$

Proof

Proof Continued.

Conversely, suppose that $U = K \cap V$. Then if $x \in U$, we have $x \in V$ and there is a $r > 0$ such that $B_r(x) \subset V$. But then $B_r^K(x) = B_r(x) \cap K \subset V \cap K = U$. This shows U is open in K. $\square$

Proposition

Suppose that K is a subspace of E. Then K is a compact subset of E is and only if K is compact (as a subset of itself).

Proof.

Suppose that K is a compact subset of E. Suppose $K = \bigcup_{i \in I} U_i$ with each U_i an open subset of K. Then there are open sets V_i in E such that $U_i = K \cap V_i$. But then $K \subset \bigcup_i V_i$ and there is a finite set $F \subset I$ such that $K \subset \bigcup_{i \in F} V_i$. But then $K = \bigcup_{i \in F} U_i$ and K is a compact metric space.

Proof

Proof Continued.

Conversely, suppose K is a compact metric space. Suppose that $K \subset \bigcup_{i \in I} V_i$ with V_i open in E. Then $K = \bigcup_{i \in I} K \cap V_i$. Since each $K \cap V_i$ is open in K, there is a finite set $F \subset I$ such that $K = \bigcup_{i \in F} K \cap V_i$. But then $K \subset_{i \in F} V_i$ as required. $\square$

Break Time

Time for a short break.

Proposition

A closed subset of a compact metric space is compact.

Proof.

Suppose that K is closed in E and that E is compact. Suppose $K \subset \bigcup_{i\in I} V_i$ with each V_i open in E. Then $E = \mathscr{C}K \cup \bigcup_{i\in I} V_i$. Since $\mathscr{C}K$ is open, there is a finite set $F \subset I$ such that $E = \mathscr{C}K \cup \bigcup_{i\in F} V_i$. But then $K \subset \bigcup_{i\in F} V_i$. $\square$

More Fruit

Proposition

A compact subset of a metric space is bounded. In particular, any compact metric space must be bounded.

Proof.

Suppose that $K \subset E$ is compact. Then $K \subset \bigcup_{x \in K} B_1(x)$. Thus K is contained in a finite union of balls. We agreed last time that this forced K to be bounded. $\square$

Theorem (Nested Set Property)

Let E be a compact metric space. Suppose that $\{ F_n : n \in \mathbf{N} \}$ is sequence of nonempty closed sets in E such that $F_{n+1} \subset F_n$. Then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof.

Suppose to the contrary that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then, taking complements, $E = \bigcup_{n=1}^{\infty} \mathscr{C} F_n$. Since E is compact, there is a $N \in \mathbf{N}$ such that $E = \bigcup_{n=1}^{N} \mathscr{C} F_n$. Since the F_n are nested, this forces $E = \mathscr{C} F_N$ and $F_N = \emptyset$. This contradicts our assumptions on the F_n and completes the proof. □

Compactness and Closed are Essential

Remark

Note that $F_n = [n, \infty)$ is closed and nonempty in $\mathbf{R}$ for all $n \in \mathbf{N}$. But $\bigcap_n [n, \infty) = \emptyset$! This shows that compactness is essential for the nested set property and/or that $\mathbf{R}$ is not compact. In due course, we will see that $[0, 1]$ is compact (as a subspace of $\mathbf{R}$). Then $U_n = (0, \frac{1}{n})$ is open and $U_{n+1} \subset U_n$. Nevertheless, $\bigcap_n U_n = \emptyset$.

Break Time

Time or a break and a few questions.

Cluster Points

Definition

Let S be a subset of a metric space E. We say that $x \in E$ is a cluster point of S if $B_r(x) \cap S$ is infinite for all $r > 0$.

Example

Let $E = \mathbf{R}$ and $S = \{ \frac{1}{n} : n \in \mathbf{N} \}$. Then 0 is the only cluster point of S.

Theorem

Suppose that E is a compact metric space. Then every infinite subset of E has at least one cluster point.

Proof

Proof.

Suppose to the contrary S is an infinite subset of E with no cluster points in E. Then given $x \in E$, there is a an open ball U_x centered at x such that $U_x \cap S$ is finite. Since $E = \bigcup_{x \in E} U_x$ and E is compact, then are $x_1, \dots, x_n$ in E such that $E = \bigcup_{k=1}^n U_{x_k}$. But then $S = S \cap E = \bigcup_{k=1}^n U_x \cap S$ would be finite. □

Sequential Compactness

Definition

A metric space is called sequentially compact if every sequence has a convergent subsequence.

Theorem

A compact metric space is sequentially compact.

Proof.

Let (x_n) be a sequence in a compact metric space E.

Case I: Suppose that $S = \{ x_n : n \in \mathbf{N} \}$ is infinite. Then S has a cluster point $x \in E$. Then $B_{\frac{1}{n}}(x) \cap S$ is infinite for every $n \in \mathbf{N}$. Find n_1 such that $x_{n_1} \in B_1(x)$. Since $B_{\frac{1}{2}}(x) \cap S$ is infinite, we can find $n_2 > n_1$ such that $x_{n_2} \in B_{\frac{1}{2}}(x)$. If we have found $n_1 < n_2 < \cdots < n_k$ such that $x_{n_j} \in B_{\frac{1}{j}}(x)$, then since $B_{\frac{1}{k+1}}(x) \cap S$ is infinite, we can find $n_{k+1} > n_k$ with $x_{n_{k+1}} \in B_{\frac{1}{k+1}}(x)$. Thus we get a subsequence (x_{n_k}) that clearly converges to x.

Proof

Proof Continued.

Case II: Otherwise $\{ x_n : n \in \mathbf{N} \}$ is finite. Then for some $x \in \{ x_n : n \in \mathbf{N} \}$, then $\{ n \in \mathbf{N} : x_n = x \}$ is infinite. Proceeding as above, we can find $n_1 < n_2 < \dots$ such that $x_{n_k} = x$ for all k. Then (x_{n_k}) is the constant sequence and converges to x. □

Theorem

A compact metric space is complete.

Proof.

Suppose that (x_n) is a Cauchy sequence in a compact metric space E. Since E is sequentially compact, (x_n) has a convergent subsequence. Hence (x_n) converges and E is complete. □

Closed Subsets

Proposition

If K is a compact subset of a metric space E, then K is a closed subset of E.

Proof.

Suppose that (x_n) is a sequence in K that converges to some $x \in E$. It will suffice to see that $x \in K$. But since (x_n) converges, it must be Cauchy. Since K is compact, it is complete, and (x_n) must converge to some $y \in K$ and therefore in E as well. Since limits are unique, $x = y \in K$. □

Enough

1. That is enough for today.