Math 63: Winter 2021
Lecture 9

Dana P. Williams

Dartmouth College

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Getting Started

1. We should be recording.
2. Our preliminary exam will be available on `gradescope` after class on Friday, January 29th, and must be turned in by Sunday January 31st by 10pm. You will have 150 minutes to work the exam and an additional 30 minutes to scan, link, and upload your exam to `gradescope`. The Exam will cover through §III.5 in the text most of which I hope to finish today.
3. I suggest blocking out a three hour window now so that you can work undisturbed. Although officially a take home exam, it is effectively an "in class" exam where you get to pick when you take it.
4. Time for some questions!

Review

Definition

A subset K of a metric space E is said to be compact if, given any collection $\{ U_i : i \in I \}$ of open sets in E such that $K \subset \bigcup_{i \in I} U_i$, then there is a finite subset $F \subset I$ such that $K \subset \bigcup_{i \in F} U_i$.

Proposition

Suppose that K is a subspace of E. Then K is a compact subset of E is and only if K is compact (as a subset of itself).

Definition

Let S be a subset of a metric space E. We say that $x \in E$ is a cluster point of S if $B_r(x) \cap S$ is infinite for all $r > 0$.

Review

Theorem

Suppose that E is a compact metric space. Then every infinite subset of E has at least one cluster point.

Definition

A metric space is called sequentially compact if every sequence has a convergent subsequence.

Theorem

A compact metric space is sequentially compact.

Theorem

A compact metric space is complete.

The Big Theorem

Theorem

Suppose $N \in \mathbf{N}$. A subset S of Euclidean space E^N is compact if and only if it is closed and bounded.

Remark

This result will take some work to prove and we will start by collecting some preliminary results.

Lemma

If $K \subset E^N$ is compact, then K is closed and bounded.

Proof.

This is true in any metric space. $\square$

Closed Intervals

Notation

If $M > 0$ and $x \in \mathbf{R}^N$ then we let $R_M(x)$ the "closed interval"

$$\begin{aligned} R_M(x) &= \{\, y \in E^N : |y_k - x_k| \leq M \text{ for all } 1 \leq k \leq N \,\} \\ &= \{\, y \in E^N : x_k - M \leq y_k \leq x_k + M \text{ for all } 1 \leq k \leq N \,\} \end{aligned}$$

We will write $CB_M(x) = \{\, y \in E^N : d(y, x) \leq M \,\}$ for the closed ball of radius M centered at x.

Lemma

For all $M > 0$, we have

$$R_{\frac{M}{\sqrt{N}}}(x) \subset CB_M(x) \subset R_M(x).$$

Proof

Proof.

Since we always have $|y_k - x_k| \leq d(y,x)$, the second containment is clear. So suppose $y \in R_{\frac{M}{\sqrt{N}}}(x)$. Then

$$d(y,x)^2 = \sum_{k=1}^{N} (y_k - x_k)^2 \leq \sum_{k=1}^{N} \frac{M^2}{N} = M^2.$$

Therefore $d(y,x) \leq M$ and $y \in CB_M(x)$. □

Bounded Sets and Balls

Proposition

Suppose that S is a bounded subset of E^N and $\epsilon > 0$. Then S is covered by finitely many closed ϵ-balls.

Remark

This is somewhat special to Euclidean space. Let $E = \mathbf{N}$ with the discrete metric. Then E is bounded. But E cannot be covered by finitely many $\frac{1}{2}$-balls!

Proof.

Since S is bounded, there is a $M > 0$ such that $S \subset B_M(0)$ (where here 0 is the zero vector $(0, 0, \dots, 0)$ in E^N). Then $S \subset R_M(0)$ by our lemma. So it suffices to prove the proposition for the closed interval $R_M(0)$.

Proof of Proposition

Proof Continued.

Choose k such that $\frac{1}{k} < \frac{\epsilon}{\sqrt{N}}$. Let $a_m = -M + \frac{m}{k}$ for $m \in 0, 1, \ldots, 2kM$. In fancy language, $P_k = \{\, a_m : 0 \leq m \leq 2kM \,\}$ forms a regular partition of $[-M, M]$ into subintervals of length $\frac{1}{k}$. Let $P_k^N = \{\, x \in E^N : x_n \in P_k \text{ for } 1 \leq n \leq N \,\}$. (Draw a picture in E^2.) Note that P_k^N is finite. Furthermore, $R_M(0) \subset \bigcup_{x \in P_k^N} R_{\frac{1}{k}}(x)$. But our lemma implies $R_{\frac{1}{k}}(x) \subset CB_{\frac{\sqrt{N}}{k}}(x) \subset CB_\epsilon(x)$. Therefore,

$$S \subset \bigcup_{x \in P_k^N} CB_\epsilon(x). \qquad \square$$

Proof of the Theorem.

Suppose that S is a closed and bounded subset of E^N. Suppose to the contrary of what we want to prove, that S is not compact. Then there are open sets $\{ U_i \}_{i \in I}$ in E^N such that $S \subset \bigcup_{i \in I} U_i$ and no finite subset of $\{ U_i \}_{i \in I}$ covers S.

Using the proposition, we can cover S by finitely many closed $\frac{1}{2}$-balls—say, $B_1, \dots, B_r$. Then

$$S = \bigcup_{k=1}^{r} S \cap B_k.$$

Then at least one of the sets $S \cap B_k$ is not covered by finitely many U_i. Let S_1 be that set. Note that S_1 is closed, bounded, and is not covered by finitely many U_i. Furthermore, if $x, y \in S_1$, then $d(x, y) \leq 1$ (Why?)

Proof

Proof Continued.

Now we apply the proposition to S_1 and cover it with finitely many $\frac{1}{4}$-balls. Repeating the above argument, we get a closed bounded subset $S_2 \subset S_1$ that can't be covered by finitely many U_i such that $x, y \in S_2$ implies $d(x, y) \leq \frac{1}{2}$. Continuing in this manner, using balls of radius $\frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots, \frac{1}{2n}, \dots$, we get closed subsets $S_{n+1} \subset S_n$ such that no S_n can be covered by finitely many U_i and $x, y \in S_n$ implies that $d(x, y) \leq \frac{1}{n}$. Clearly, no S_n can be empty, so we can pick $x_n \in S_n$. I claim that (x_n) is Cauchy. If $\epsilon > 0$, there is a K such that $\frac{1}{K} < \epsilon$. Then if $n, m \geq K$, we have $x_n, x_m \in S_K$ and $d(x_n, x_m) \leq \frac{1}{K} < \epsilon$. Thus (x_n) is Cauchy as claimed.

Proof

Proof Continued.

Since E^N is complete, we can assume there is an x_0 such that $x_n \to x_0$. Since all the $x_n \in S$ and S is closed, we have $x_0 \in S$. Since the U_i cover S, there is an i_0 such that $x_0 \in U_{i_0}$. Since U_{i_0} is open, there is a $\epsilon > 0$ such that $B_\epsilon(x_0) \subset U_{i_0}$. Let k be such that $\frac{1}{k} < \frac{\epsilon}{2}$ and $d(x_k, x_0) < \frac{\epsilon}{2}$. Now if $x \in S_k$, we have

$$d(x, x_0) \leq d(x, x_k) + d(x_k, x_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $S_k \subset U_{i_0}$. This contradicts our assumption that no S_n can be covered by finitely many U_i. $\square$

Break Time

Time for a well-deserved break!! Also a question or three.

Remark

We come to another important topological concept which, similar to the definition of compactness, is not very intuitive. Namely, we want to define what it means for a metric space to be "connected". Unfortunately, even in $\mathbf{R}$ or E^2 it is not so clear what it means to be "just one piece"—or whatever the English definition of connected is—into precise mathematical terms. As with compactness, the definition will be validated from the results.

Definition

A metric space E is connected if the only subsets of E that are both open and closed are E and $\emptyset$. We say that a subset S if E is connected if is connected as a subspace.

Open and Closed Partitions

Remark

If E is not connected, then there is a nonempty proper subset $A \subset E$ which is both open and closed. Then $B = \mathscr{C}A$ is also a proper nonempty subset which is both open and closed. Then $E = A \cup B$ and $A \cap B = \emptyset$. We call $E = U \cup V$ a partition of E if $U \cap V = \emptyset$. If we can find a partition $E = A \cup B$ with both A and B open and nonempty, then A is a nonempty closed and open proper subset. Hence E is not connected. We can also work with closed partitions.

Between

Remark

If $a, b, c \in \mathbf{R}$, then we say that c lies between a and b if either $c \in (a, b)$ or $c \in (b, a)$.

Proposition

Suppose that S is a subset of $\mathbf{R}$ that contains two distinct points a and b. If S connected, then S contains all the points between a and b.

Proof.

We may as well assume that $a < b$. Suppose to the contrary that for some $c \in (a, b)$, we have $c \notin S$. Then

$$S = [(-\infty, c) \cap S] \cup [(c, \infty) \cap S]$$

is a partition of S into two open (in S) nonempty sets. This contradicts the assumption that S is connected. $\square$

Unions of Connected Sets

Proposition

Suppose that $\{ S_i : i \in I \}$ is a collection of connected sets in a metric space E. Suppose that for some $i_0 \in I$, we have $S_{i_0} \cap S_i \neq \emptyset$ for all $i \in I$. Then $S = \bigcup_{i \in I} S_i$ is connected.

Proof.

Note that each S_i must be nonempty. Suppose that S is the union of disjoint open sets A and B. We have to show that either A or B must be empty. Then for each i, $S_i = (A \cap S_i) \cup (B \cap S_i)$. Since since both these sets are open in S_i and disjoint, one of them must be empty and the other all of S_i. In particular, swapping A and B if necessary, we an assume $S_{i_0} \subset A$. But then $A \cap S_i \neq \emptyset$ for all i and $S_i \subset A$ for all i. Thus $S \subset A$ and B is empty. □

Theorem

Let S be a subset of $\mathbf{R}$ that contains all the points between any two points in S. Then S is connected.

Proof.

Suppose to the contrary that S is not connected. Then and we can find a partition $S = A \cup B$ with A and B both nonempty and open. Let $a \in A$ and $b \in B$. We can assume $a < b$. Then by assumption, $[a, b] \subset S$. Let $A_1 = A \cap [a, b]$ and $B_1 = B \cap [a, b]$. Since $a \in A_1$ and $b \in B_1$, $[a, b] = A_1 \cup B_1$ is a partition of $[a, b]$ into open subsets. Then A_1 and B_1 are also closed subsets of $[a, b]$. Since A_1 is closed and bounded above, it contains a largest element c. Since $b \in B_1$, $c < b$. But A_1 is open, so it contains an open interval containing c. Thus A_1 contains elments larger than c. This is a contradiction. □

Intervals

Corollary

The real line, $\mathbf{R}$ *is connected as are any intervals be they open, closed, or half-open.*

Enough

1. That is enough for today.