Math 63: Winter 2021
Lecture 10

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Friday, January 29, 2021

Getting Started

1. We should be recording.
2. Our preliminary exam will be available on `gradescope` after class. It must be turned in by Sunday January 31st by 10pm. You will have 150 minutes to work the exam and an additional 30 minutes to scan, link, and upload your exam to `gradescope`. The Exam will cover through §III.5 in the text.
3. I suggest blocking out a three hour window now so that you can work undisturbed. Although officially a take home exam, it is effectively an "in class" exam where you get to pick when you take it.
4. Time for some questions!

Notation

If (E, d) and (E', d') are metric spaces, then we write $f : (E, d) \to (E', d')$ to denote a function $f : E \to E'$ and at the same time name the metric d on E and d' on E'. Just as previously, we will often simple write $f : E \to E'$ and name the metrics later.

Definition

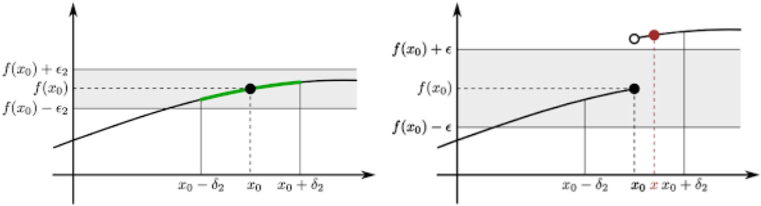
A function $f : (E, d) \to (E', d')$ is said to be continuous at $x_0 \in E$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$.

Remark

Formally, the δ is the above definition depends on the choice of both $\epsilon > 0$ and $x_0 \in E$. But we get fussy and write $\delta = \delta(\epsilon, x_0)$ when we need to emphasize this point.

Continuity on the Real Line

If $f : \mathbf{R} \to \mathbf{R}$, then continuity at x_0 reduces to the usual "ϵ–δ-definition" from back in the day: namely, given $x_0 \in \mathbf{R}$ and $\epsilon > 0$ there is a $\delta > 0$ so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.



On the left is a picture of function $f : \mathbf{R} \to \mathbf{R}$ that is continuous at x_0 and on the right of a similar function that fails to be continuous at x_0. (I stole the artwork from the internet, so I have no idea why the ϵ's and δ's are decorated with subscripts.)

Alternatives

Remark

A simple reformulation of the definition is that $f:(E,d)\to(E',d')$ is continuous at $x_0\in E$ if and only if given $\epsilon>0$ there is a $\delta>0$ such that

$$f(B^E_\delta(x_0))\subset B^{E'}_\epsilon(f(x_0)) \iff B^E_\delta(x_0)\subset f^{-1}\big(B^{E'}_\epsilon(f(x_0))\big).$$

Definition

Let W be a subset of a metric space a neighborhood of $x_0\in E$ if W contains an open subset containing x_0.

Lemma

A function $f:E\to E'$ is continuous at x_0 if $f^{-1}(V)$ is a neighborhood of x_0 whenever V is a neighborhood of $f(x_0)$.

Proof.

I will leave this as an exercise. $\square$

Continuous on E

Definition

If E and E' are metric spaces, then we say that the function $f : E \to E'$ is continuous if f is continuous at every $x \in E$.

Example

Consider $f : (0, \infty) \to (0, \infty)$ given by $f(x) = \frac{1}{x}$. Show that f is continuous.

Solution

Fix $x_0 \in (0, \infty)$. *Let* $\delta = \min\{ \frac{x_0}{2}, \epsilon \frac{x_0^2}{2} \}$. *Then if* $|x - x_0| < \delta$, *we have* $x = x_0 - (x_0 - x) \geq x_0 - |x_0 - x| \geq \frac{x_0}{2} > 0$. *Then*

$$|f(x) - f(x_0)| = \frac{|x - x_0|}{|x x_0|} \leq |x - x_0| \frac{2}{x_0^2} < \epsilon.$$

Thus f *is continuous as claimed.*

Example

Example

Let (E, d) be a metric space and $p_0 \in E$. Let $f(x) = d(x, p_0)$. Then f is continuous from E to $\mathbf{R}$.

Solution

Note that $|f(x) - f(y)| = |d(x, p_0) - d(y, p_0)| \leq d(x, y)$ *by the reverse triangle inequality. Hence given* $x_0 \in E$ *and* $\epsilon > 0$ *we can let* $\delta = \epsilon$. *Then* $|f(x) - f(x_0)| \leq d(x, x_0) < \epsilon$. *Note that in this case, our choice of* δ *did not depend on the choice of* x_0.

Example

Let E be a metric space with $x_0 \in E$. Let $f : E \to E'$ be the constant function $f(x) = x_0 \in E'$ for all x. Then f is continuous.

More Examples

Example

Let E be a metric space and $f : E \to E$ the identity function; that is, $f(x) = x$ for all $x \in E$. Then f is continuous. In the case where $E = \mathbf{R}$, then we usually denote this function by simply x.

Example

Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational, and} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at x_0 if and only if $x_0 = 0$.

Solution

Solution

Suppose $x_0 = 0$ *and* $\epsilon > 0$. *Since* $|f(x) - f(x_0)| = |x - 0|$ *we can take* $\delta = \epsilon$ *and* f *is continuous at* 0. *Now suppose* $x_0 > 0$ *and rational. Let* $\epsilon = x_0$. *Then for all* $\delta > 0$, *there is a irrrational* x *such that* $|x - x_0| < \min\{\,\delta, |x_0|\,\}$. *Then* $x > 0$ *and since* x *is irrational* $|f(x) - f(x_0)| = x + x_0 > x_0$. *Therefore* f *is not continuous at* x_0. *The proof is similar if* $x_0 > 0$ *and irrational, or if* $x_0 < 0$.

Restrictions

Remark

Suppose that $f : E \to E'$ is continuous. Suppose that S is a subspace of E and $f|_S : S \to E'$ the restriction of f to S. Then $f|_S$ is continuous.

Solution

This is practically automatic. Suppose $x_0 \in S$ and $\epsilon > 0$. Then there is a $\delta > 0$ so that if $x \in E$ and $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \epsilon$. But then if $y \in S$ and $d(y, x_0) < \delta$, then
$d'(f|_S(y), f(x_0)) = d'(f(y), f(x_0)) < \epsilon$ as required.

Remark

Similarly, if S' is a subspace of E' and if $f : E \to S'$ a function, we can formally extend f to a function $f'' : E \to E'$ by letting $f''(x) = f(x)$. Then f is continuous if and only if f'' is continuous. Then for example, the function $f(x) = \frac{1}{x}$ is continuous from $(0, 1)$ to $(1, \infty)$.

Break Time

Time for a quick break and questions.

Topological Continuity

Theorem

Suppose that E and E' are metric spaces and $f : E \to E'$ is a function. Then f is continuous if and only if $f^{-1}(V) = \{\, x \in E : f(x) \in V \,\}$ is open in E whenever V is open in E'.

Proof.

Suppose that $f : E \to E'$ is continuous. Let V be open in E' and fix $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$ and there is a $\epsilon > 0$ such that $B_\epsilon^{E'}(f(x_0)) \subset V$. Since f is continuous at x_0, there is a $\delta > 0$ such that $f(B_\delta^E(x_0)) \subset B_\epsilon^{E'}(f(x_0))$. Hence

$$B_\delta^E(x_0) \subset f^{-1}\big(B_\epsilon^{E'}(f(x_0)\big) \subset f^{-1}(V).$$

Since $x_0 \in f^{-1}(V)$ was arbitrary, this shows $f^{-1}(V)$ is open.

Proof

Proof Continued.

Conversely, now suppose that the inverse image of open sets is open. Let $x_0 \in E$ and $\epsilon > 0$. Since $B_\epsilon^{E'}(f(x_0))$ is open in E', by assumption, $f^{-1}\big(B_\epsilon^{E'}(f(x_0))\big)$ is open in E. Since $x_0 \in f^{-1}\big(B_\epsilon^{E'}(f(x_0))\big)$, there is a $\delta > 0$ such that $B_\delta^E(x_0) \subset f^{-1}\big(B_\epsilon^{E'}(f(x_0))\big)$. But then

$$f\big(B_\delta^E(x_0)\big) \subset B_\epsilon^{E'}(f(x_0))$$

and we agreed that, since $\epsilon > 0$ was arbitrary, this shows f is continuous at x_0. Since x_0 was arbitrary, we've shown f is continuous as required. □

Example

Example

If $f : E \to \mathbf{R}$ is continuous, then $\{ x \in E : f(x) > a \}$ is open in E; it is the inverse image of the open set (a, ∞). Similarly, $\{ x \in E : f(x) < b \}$ and $\{ x \in E : a < f(x) < b \}$ are open in E.

Composition

Proposition

Suppose that $f : (E, d) \to (E', d')$ *and* $g : (E', d') \to (E'', d'')$ *are functions. Then we can form the composition* $g \circ f : (E, d') \to (E'', d'')$ *(recall that* $g \circ f(x) := g\big(f(x)\big)$*). If* f *is continuous at* x_0 *and* g *is continuous at* $f(x_0)$ *then* $g \circ f$ *is continuous at* x_0*. In particular, if both* f *and* g *are continuous, then so is* $g \circ f$*.*

Remark

In flowery terms, "the composition of continuous functions is continuous".

Proof

Proof.

Fix x_0 and $\epsilon > 0$. Since g is continuous at $f(x_0)$ there is a δ' such that $d'(y, f(x_0)) < \delta'$ implies $d''(g(y), g(f(x_0))) < \epsilon$. Since f is continuous at x_0, there is a $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \delta'$. But then $d''(g(f(x)), g(f(x_0))) < \epsilon$. This shows that $g \circ f$ is continuous at x_0. Now the second assertion follows as well. □

Remark

If we just want to prove that $g \circ f$ is continuous when f and g are, we can use our open set criterion. If V is open in E'', then $g \circ f^{-1}(V) = f^{-1}(g^{-1}(V))$. Then $g^{-1}(V)$ is open in E', so that $f^{-1}(g^{-1}(V))$ is open in E as required.

A Bit of Fluff

Example

Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational, and} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \in \mathbf{Z},\ q \in \mathbf{N} \text{ with } (p,q) = 1 \end{cases}$$

Here $(p, q) = 1$ just means that p and q have no nontrivial common factors. This function is sometimes called the ruler function. Why do you suppose that is? Does anyone want to speculate on where, if anywhere, the ruler function is continuous?

Solution

I claim that the ruler function is continuous at every irrational and discontinuous at every rational. Suppose that x is rational and that $f(x) = \frac{1}{q}$. Let $0 < \epsilon < \frac{1}{q}$. Then for any $\delta > 0$, there are irrational y such that $|y - x| < \delta$. But then $|f(y) - f(x)| = \frac{1}{q} > \epsilon$. So f is not continuous at x.

Now suppose that x is irrational and $\epsilon > 0$. Then there is a $n \in \mathbf{Z}$ such that $n < x < n + 1$. Since there is a $N \in \mathbf{N}$ such that $\frac{1}{N} < \epsilon$, there are only finitely many $q \in \mathbf{N}$ such that $\frac{1}{q} \geq \epsilon$. But if $p \in \mathbf{Z}$ is such that $n \leq \frac{p}{q} \leq n + 1$ then $qn \leq p \leq q(n + 1)$. Hence for any $q \in \mathbf{N}$, there are only finitely many rationals of the form $\frac{p}{q}$ in $[n, n + 1]$. Thus the set F of $r \in [n, n + 1]$ such that $f(x) \geq \epsilon$ is finite. We can assume $n, n + 1 \in F$. Let $\delta = \min\{\, |r - x| : r \in F \,\}$. Then if $|y - x| < \delta$, we have $|f(y) - f(x)| = f(y) < \epsilon$. Thus the ruler function is continuous at every irrational.

Enough

1. That is enough for today.