Math 63: Winter 2021
Lecture 12

Dana P. Williams

Dartmouth College

Wednesday, February 3, 2021

Getting Started

1. We should be recording.
2. Time for some questions!

Review

Theorem

Suppose that f and g are real-valued functions on E that are continuous at x_0. Then $f+g$, $f-g$ and fg are also continuous at x_0. If in addition, $g(x_0) \neq 0$, then the quotient, $\frac{f}{g}$, is continuous (on its natural domain) at x_0 as well.

Review

Theorem

Suppose x_0 is a cluster point of E and that f and g are real-valued functions on $\mathscr{C}\{x_0\}$ such that

$$\lim_{x\to x_0} f(x) = L \quad \textit{and} \quad \lim_{x\to x_0} g(x) = M.$$

Then

$$\lim_{x\to x_0} f(x) \pm g(x) = L \pm M \quad \textit{and} \quad \lim_{x\to x_0} f(x)g(x) = LM.$$

If $M \neq 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Coordinate Functions

Proposition

Define $X_k : E^n \to \mathbf{R}$ by $X_k(x_1, \dots, x_n) = x_k$. We call the X_k the coordinate functions. Then for all $1 \le k \le n$, X_k is continuous.

Proof.

The notation is a bit cumbersome. If $p \in E^n$, then $p = \big(X_1(p), \dots, X_n(p)\big)$. Hence

$$|X_k(p) - X_k(p_0)|^2 \le \sum_{j=1}^{n} (X_j(p) - X_j(p_0))^2 = d(p, p_0)^2.$$

Hence given $\epsilon > 0$, we can let $\delta = \epsilon$, and then $d(p, p_0) < \delta$ implies $|X_k(p) - X_k(p_0)| < \epsilon$. That is, X_k is continuous. □

Nice Functions are Continuous

Example

It now follows easily from the previous result and the last lecture that any polynomial p in the variables $x_1, \dots, x_n$ is continuous on E^n. For example, $p(x, y, z) = xy^2z^3 + 27xy + z + 7$ is continuous on E^3. Similarly, any rational function—that is, the quotient of two polynomials—is continuous on its natural domain. Thus $f(x, y) = \frac{xy}{x^2+y^2}$ is continuous on $E^2 \setminus \{(0, 0)\}$.

Lemma (HW)

The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example

Since the composition of continuous function is continuous, we have $g(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$ continuous on $E^2 \setminus \{\,(0, 0)\,\}$.

Example

Example

Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

Solution

Note that $|x|$ *and* $|y|$ *are both bounded by* $\sqrt{x^2+y^2}$. *Also, observe that if* $(x,y)\neq(0,0)$, *then*

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = |x|\left|\frac{y}{\sqrt{x^2+y^2}}\right| \leq |x|.$$

Then given $\epsilon>0$, *we can let* $\delta=\epsilon$ *and then* $0<d((x,y),(0,0))<\delta$ *implies* $|g(x,y)-0|<\epsilon$. *Therefore the limit exists and equals* 0. *Therefore, the function*

$$h(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0), \text{ and}\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

is continuous on E^2.

Break Time

Time for a break and some questions.

Component Functions

Definition

If $f : E \to E^n$ is a function, then in our awkward notation, $f(p) = \big(X_1(f(p)), \dots, X_n(f(p))\big)$. The real-valued functions $X_k \circ f$ are called the component functions of f. Conversely, given functions $g_k : E \to \mathbf{R}$ for $1 \le k \le n$, we can build a function $f : E \to E^n$ by $f(p) = \big(g_1(p), \dots, g_n(p)\big)$ and $X_k \circ f = g_k$.

Proposition

A function $f : E \to E^n$ is continuous at $p_0 \in E$ if and only if all its component functions $X_k \circ f$ are continuous at p_0.

Proof.

If $f : E \to E^n$ is continuous, then $X_k \circ f$ is continuous since X_k is continuous at $f(p_0)$ (since it is continuous everywhere).

Proof Continued.

Conversely, suppose $X_k \circ f$ is continuous at p_0 for $1 \le k \le n$. Fix $\epsilon > 0$. Let $\delta_k > 0$ be such that $d^E(p, p_0) < \delta_k$ implies $|X_k(f(p)) - X_k(f(p_0))| < \frac{\epsilon}{\sqrt{n}}$. Let $\delta = \min\{\, \delta_1, \dots, \delta_n \,\}$. Then $\delta > 0$ and $d^E(p, p_0) < \delta$ implies

$$\begin{aligned} d(f(p), f(p_0))^2 &= \sum_{k=1}^{n} \big[X_k(f(p)) - X_k(f(p_0))\big]^2 \\ &< \sum_{k=1}^{n} \frac{\epsilon^2}{n} = \epsilon^2 \end{aligned}$$

and $d(f(p), f(p_0)) < \epsilon$. Thus f is continuous at p_0 as claimed. □

Limits Again

Example

Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

Solution

Let $h(x, y) = \frac{x^2-y^2}{x^2+y^2}$*. Suppose that the limit did exist and was equal to* L*. Then we could extend* h *to a continuous function on* E^2 *by setting* $h(0, 0) = L$*. Then* $f(x) = h(x, 0)$ *would be continuous and* $\lim_{x\to 0} f(x) = f(0) = h(0, 0) = L$*. Since* $f(x) = 1$ *for all* $x \neq 0$*, we would have to have* $L = 1$*. But we could make the same analysis for* $g(y) = h(0, y)$*. Now we would have to have* $L = -1$*. Hence the limit does not exist.*

A Puzzle

Example

Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

Solution

Let $h(x,y) = \frac{x^2y}{x^4+y^2}$*. Now if we approach the origin along either coordinate axis, we get the limit* 0*. Let's approach the origin along the line* $y = mx$ *and consider* $f(x) = h(x, mx) = \frac{mx^3}{x^4+m^2x^2} = \frac{mx}{x^2+m^2}$*. Then* $|f(x)| \leq |x||m|^{-1}$ *and* $\lim_{x\to 0} f(x) = 0$*. So is the limit above equal to* 0*? Consider* $g(x) = h(x, x^2)$*. Then* $\lim_{x\to 0} g(x) = \frac{1}{2}$*! So the limit doesn't exist.*

Break Time

Time for another break and a few questions.

Compactness Again

Theorem

Suppose that E and E' are metric spaces and that $f : E \to E'$ is continuous. If E is compact, then so is its image $f(E)$.

Proof.

Let $\{ U_i \}_{i \in I}$ be a collection of open sets in E' such that

$$f(E) \subset \bigcup_{i \in I} U_i.$$

Then

$$E = \bigcup_{i \in I} f^{-1}(U_i).$$

Proof

Proof Continued.

Since f is continuous, $f^{-1}(U_i)$ is open for each $i \in I$. That is, $\{ f^{-1}(U_i) \}_{i\in I}$ is an open cover of E. Since E is compact, there is a finite set $F \subset I$ such that

$$E = \bigcup_{i\in F} f^{-1}(U_i).$$

But then

$$f(E) = \bigcup_{i\in F} f\big(f^{-1}(U_i)\big) \subset \bigcup_{i\in F} U_i.$$

Since we started with an arbitrary open over of $f(E)$, this shows $f(E)$ is compact. □

Bounded Functions

Definition

A function $f : E \to E'$ is called bounded if $f(E)$ is bounded in E'.

Remark

A real-valued function $f : E \to \mathbf{R}$ is bounded exactly when we can find a $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

Compactness is Necessary

Proposition

Suppose that $f : E \to E'$ is continuous and that E is compact. Then f is bounded.

Proof.

We know that $f(E)$ is compact in E' and compact sets are bounded in any metric space. $\square$

Example

Let $f : (0,1) \to \mathbf{R}$ be given by $f(x) = \frac{1}{x}$. Then f is not bounded.

Theorem (Extreme Value Theorem)

Suppose that $f : E \to \mathbf{R}$ is a continuous real-valued function and that E is compact. Then f attains its maximum and minimum on E. That is, there are points $p, q \in E$ such that

$$f(p) \leq f(x) \leq f(q) \quad \textit{for all } x \in E.$$

Proof.

Since $f(E) \subset \mathbf{R}$ is compact, it is closed and bounded. Hence $f(E)$ has a largest element M and a smallest element m. Then there is a $p \in E$ such that $f(p) = m$ and a $q \in E$ such that $f(q) = M$. □

Remark

Note that the function $f(x) = x$ is continuous and bounded on the open interval $(0, 1)$. But it does not attain its maximum or minimum on $(0, 1)$.

Enough

1. That is enough for today.