Math 63: Winter 2021
Lecture 13

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Getting Started

1. We should be recording.
2. Time for some questions!

Review

Theorem

Suppose that E and E' are metric spaces and that $f : E \to E'$ is continuous. If E is compact, then so is its image $f(E)$.

Definition

A function $f : E \to E'$ is called bounded if $f(E)$ is bounded in E'.

Proposition

Suppose that $f : E \to E'$ is continuous and that E is compact. Then f is bounded.

Theorem (Extreme Value Theorem)

Suppose that $f : E \to \mathbf{R}$ is a continuous real-valued function and that E is compact. Then f attains its maximum and minimum on E. That is, there are points $p, q \in E$ such that

$$f(p) \le f(x) \le f(q) \quad \text{for all } x \in E.$$

Remark

If, as usual, E and E' are metric spaces, then $f : E \to E'$ is continuous at $x_0 \in E$ if for all $\epsilon > 0$ there is a $\delta > 0$ so that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$. As we have seen in examples, we get to know what ϵ and x_0 are when we find δ. Thus δ is really a function of both ϵ and x_0. For a good example, review our proof of the continuity of $f : (0, \infty) \to (0, \infty)$ where given $\epsilon > 0$ we chose

$$\delta = \min\Big\{ \frac{x_0}{2}, \epsilon\frac{x_0^2}{2} \Big\}.$$

Definition

If E and E' are metric spaces, then we say that $f : E \to E'$ is uniformly continuous if for all $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$ for all $x, y \in E$.

Low Hanging Fruit

Remark

The point of the definition is that given ϵ, our δ has to work for all $x, y \in E$.

Lemma

If $f : E \to E'$ is uniformly continuous, then f is continuous.

Proof.

Convince yourself that this is easy. □

Remark

If $f : E \to E'$ is a function and S is a subset of E, then we say that f is uniformly continuous on S if the restriction $f : S \to E'$ is uniformly continuous. Thus f is uniformly continuous on S if for all $\epsilon > 0$ there is a $\delta > 0$ so that $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$ for all $x, y \in S$.

Uniform Continuity is Special

Example

Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^2$. Although f is clearly continuous, it is not uniformly continuous.

Solution

We can see that this "should" to be the case with a picture. (Document Camera). Let $\epsilon = 1$. Suppose there were a $\delta > 0$ such that $|x - y| < \delta$ implies $|x^2 - y^2| < 1$. Let $x_n = n$ and $y_n = n + \frac{1}{n}$. Then $|x_n^2 - y_n^2| = |2 + \frac{1}{n^2}| > 2 > 1 = \epsilon$. Since $|x_n - y_n| = \frac{1}{n}$, we can pick n large enough so that $|x_n - y_n| = \frac{1}{n} < \delta$. This is a contradiction.

Example

Example

Let E be a metric space. Let $A \subset E$ and define $f : E \to \mathbf{R}$ by $f(x) = d(x, A) := \text{g.l.b.}\{\, d(x, y) : y \in A \,\}$. Show that f is uniformly continuous.

Solution

Let $x, y \in E$. Then

$$\begin{aligned} d(x, A) &= \text{g.l.b.}\{\, d(x, z) : z \in A \,\} \\ &\le \text{g.l.b.}\{\, d(x, y) + d(y, z) : z \in A \,\} \\ &= d(x, y) + \text{g.l.b.}\{\, d(y, z) : z \in A \,\} \\ &= d(x, y) + d(y, A). \end{aligned}$$

Therefore $d(x, A) - d(y, A) \le d(x, y)$. By symmetry, $|d(x, A) - d(y, A)| \le d(x, y)$. Therefore given $\epsilon > 0$ we can let $\delta = \epsilon$. Then if $d(x, y) < \epsilon$, we have $|f(x) - f(y)| < \epsilon$ for all $x, y \in E$.

Depends on the Space

Example

Uniform continuity depends on the space. For example, consider $f : [0,1] \to \mathbf{R}$ given by $f(x) = x^2$. Then $|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq 2|x - y|$ if $x, y \in [0,1]$. Thus given $\epsilon > 0$ we can let $\delta = \frac{\epsilon}{2}$. Then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Break Time

Time for a short break and questions

Covers of Compact Sets

Theorem

Suppose that E is a compact metric space and that $\{ U_i \}_{i \in I}$ is a collection of open subsets of E such that $E = \bigcup_{i \in I} U_i$. Then there is a $\epsilon > 0$ such that given any $x \in E$ the ϵ-ball $B_\epsilon(x)$ is contained in some U_i.

Proof.

If the result were false, then for every $n \in \mathbf{N}$, there is a $x_n \in E$ such that $B_{\frac{1}{n}}(x_n)$ is not contained in any U_i. Since E is compact, (x_n) has a convergent subsequence (x_{n_k}) converging to $x \in E$. Then $x \in U_{i_0}$ for some $i_0 \in I$. Since U_{i_0} is open, there is a $r > 0$ such that $B_r(x) \subset U_{i_0}$. Let K be such that $d(x_{n_K}, x) < \frac{r}{2}$ and $\frac{1}{K} < \frac{r}{2}$.

Proof

Proof Continued.

Since $n_K \geq K$, we also have $\frac{1}{n_K} < \frac{r}{2}$. Now if $z \in B_{\frac{1}{n_K}}(x_{n_k})$, we have

$$\begin{aligned} d(z,x) &\leq d(z,x_{n_K}) + d(x_{n_K},x) \\ &< \frac{1}{n_K} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

This implies $B_{\frac{1}{n_K}}(x_{n_K}) \subset B_r(x) \subset U_{i_0}$ and contradicts our choice of x_{n_K}. □

Compact Sets and Uniform Continuity

Theorem

Suppose E and E' are metric space and $f : E \to E'$ is continuous. If E is compact, then f is uniformly continuous.

Remark

There are two different proofs of this given in the text. Both are worth a look. We will give a proof using the previous result here.

Proof

Proof.

Fix $\epsilon > 0$. Since f is continuous, for each $x \in E$, there is a $\delta_x > 0$ be such that $d(y, x) < \delta_x$ implies $d'(f(y), f(x)) < \frac{\epsilon}{2}$. Then

$$E = \bigcup_{x \in E} B_{\delta_x}(x).$$

Now let $\epsilon > 0$. Since E is compact, there is a $\delta > 0$ so that for all $x \in E$, $B_\delta(x)$ is contained in some $B_{\delta_z}(z)$. If $d(x, y) < \delta$, then as above, for some $z \in E$, we have $B_\delta(x) \subset B_{\delta_z}(z)$. But then $y \in B_\delta(x)$ and both $d(x, z) < \delta_z$ and $d(y, z) < \delta_z$ and

$$d'(f(x), f(y)) \leq d'(f(x), f(z)) + d'(f(z), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $x, y \in E$ are arbitrary, f is uniformly continuous. □

Break Time

Time for a break and some questions.

Example

Proposition

Suppose that $f : (0,1] \to \mathbf{R}$ is continuous. Show that we can extend to a continuous function $g : [0,1] \to \mathbf{R}$ if and only if f is uniformly continuous on $(0,1]$.

Proof.

If g exists, then g is uniformly continuous since $[0,1]$ is compact. Hence f is uniformly continuous on $(0,1]$.

Now assume f is uniformly continuous. Let (x_n) be a sequence in $(0,1]$ converging to 0. Then (x_n) is Cauchy. I claim $(f(x_n))$ is Cauchy. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ if $x, y \in (0,1]$. But there is a $N \in \mathbf{N}$ such that $n, m \geq N$ implies $|x_n - y_n| < \delta$. Then $|f(x_n) - f(x_m)| < \epsilon$. Therefore $(f(x_n))$ is Cauchy, and since $\mathbf{R}$ is complete, there is a $L \in \mathbf{R}$ such that $f(x_n) \to L \in \mathbf{R}$.

Proof

Proof Continued.

Now we define $g(0) = L$ (of course, $g(x) = f(x)$ if $x \in (0,1]$). We need to see that g is continuous. Since g extends f, g is easily seen to be continuous at any $x_0 \in (0,1]$: this is because g agrees with f on an open ball centered at x_0 and hence $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f(x) = f(x_0)$. (We sometimes say that continuity is a local property.)

To show that g is continuous at 0 it suffices to show that $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = L$. Let $\epsilon > 0$. Since f is uniformly continuous we can find $\delta' > 0$ be such that $|x - y| < \delta'$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in (0,1]$. Let (x_n) be the sequence used to define L above. Then there is a N such that $|x_N - 0| < \frac{\delta'}{2} := \delta$ and $|f(x_N) - L| < \frac{\epsilon}{2}$. Then if $0 < |x - 0| < \delta = \frac{\delta'}{2}$, we have $|x - x_N| < \delta'$ and

$$|f(x) - L| \leq |f(x) - f(x_N)| + |f(x_N) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Thus $\lim_{x\to 0} f(x) = L$ and we're done. □

Some Fun

Example

The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. This an example worked out in the text. However, this follows easily from our proposition (since f trivially extends continuously to $(0, 1]$, but can't be extended continuously to 0 since $\lim_{x \to 0} f(x)$ does not exist).

Getting Nasty

Example

Let's suspend the rules for bit and assume we know the usual things about $f(x) = \sin(x)$. Then it is shown carefully in the text that $g(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, 1]$.

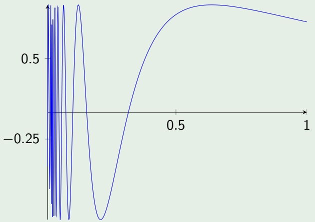

Figure: The Graph of $y = \sin\left(\frac{1}{x}\right)$

Of course, we could establish this just by showing that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

More

Example (More Interesting)

We can get a bit more funky by considering $h(x) = x \sin\left(\frac{1}{x}\right)$.

Figure: The Graph of $y = x \sin\left(\frac{1}{x}\right)$

Since $|\sin(x)| \leq 1$ for all x, we can show directly that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Thus by our proposition, $h(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1]$.

Enough

1. That is enough for today.