Math 63: Winter 2021
Lecture 16

Dana P. Williams

Dartmouth College

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Getting Started

1. We should be recording.
2. Remember it is better for me if you have your video on so that I don't feel I'm just talking to myself.
3. Our midterm will be available Friday after class and due Sunday by 10pm. It will cover through Monday's lecture. More details soon.
4. Time for some questions!

Review

Definition

Suppose that E and E' are metric spaces and that (f_n) is a sequence of functions from E to E'. We say that (f_n) converges at $p \in E$ if the sequence $(f_n(p))$ converges in E'. We say that (f_n) converges pointwise if it converges for all $p \in E$. If (f_n) converges pointwise and we let

$$f(p) = \lim_{n\to\infty} f_n(p) \quad \text{for all } p \in E,$$

the we say that (f_n) converges pointwise to f on E. We sometimes write "$f_n \to f$ pointwise on E".

Remark

We can reformulate the definition that (f_n) converge to f pointwise on E as follows. For all $p \in E$ and for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $d'(f_n(p), f(p)) < \epsilon$. Thus $N = N(\epsilon, p)$ depends on both ϵ and p.

Review

Definition

Suppose that E and E' are metric spaces and that (f_n) is a sequence of functions from E to E'. If $f : E \to E'$ is a function, then we say that (f_n) converges uniformly to f on E if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that for all $n \geq N$ we have $d'(f_n(x), f(x)) < \epsilon$ for all $x \in E$.

Definition

Let E and E' be metric spaces and (f_n) a sequence of functions from E to E'. We say that (f_n) is uniformly Cauchy if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $m, n \geq N$ implies

$$d'(f_n(x), f_m(x)) < \epsilon \quad \text{for all } x \in E.$$

Remark

If E is a metric space and $f, g : E \to \mathbf{R}$ are continuous real-valued functions, then $h(x) = |f(x) - g(x)|$ is clearly continuous: we know $f - g$ is continuous and the absolute-value function is continuous so so is their composition. It will be convenient to dress this up a little for functions mapping into a metric space E'.

Lemma

Suppose that E and E' are metric spaces and that $f, g : E \to E'$ are continuous. Then so is $h(x) = d'(f(x), g(x))$.

Remark

To prove this, it will be useful to recall the "reverse triangle inequality": $|d'(x, z) - d'(y, z)| \leq d'(x, y)$ for any metric d'.

Proof

Proof of the Lemma.

If $x, y \in E$, then

$$\begin{aligned} |h(y) - h(x)| &= |d'(f(y), g(y)) - d'(f(x), g(x))| \\ &\leq |d'(f(y), g(y)) - d'(f(x), g(y)| \\ &\qquad\qquad + |d'(f(x), g(y)) - d'(f(x), g(x))| \\ &\leq d'(f(y), f(x)) + d'(g(y), g(x)). \end{aligned}$$

Then, since f and g are continuous at x, given $\epsilon > 0$, we can choose $\delta > 0$ so that $d(y, x) < \delta$ implies both $d'(f(y), f(x)) < \frac{\epsilon}{2}$ and $d'(g(y), g(x)) < \frac{\epsilon}{2}$. But then $d(y, x) < \delta$ implies $|h(x) - h(y)| < \epsilon$ as required. □

Remark

Suppose that E and E' are metric spaces with E compact. Then if $f, g : E \to E'$ are continuous, $h(x) = d'(f(x), g(x))$ is continuous on E, and must attain its maximum. Hence we can define

$$D(f, g) = \max\{\, d'(f(x), g(x)) : x \in E \,\}.$$

We think of $D(f, g)$ as the "distance" from f to g. Some pictures are useful here.

Continuous Functions on E

Definition

Let E and E' be metric spaces with E compact. We let $C(E, E')$ be the set of all continuous functions $f : E \to E'$. If $E' = \mathbf{R}$, then we usually write $C(E)$ in place of $C(E, \mathbf{R})$.

Remark

Part of the message in the previous slide is that $D(f, g) = \max\{\, d'(f(x), g(x)) : x \in E \,\}$ defines a function $D : C(E, E') \times C(E, E') \to [0, \infty)$.

Proposition

If E and E' are metric spaces with E compact then D is a metric on E and $\big(C(E, E'), D\big)$ is a metric space.

Proof

Proof.

We have to check the three axioms for a metric. Clearly $D(f,g) = 0$ if and only if $f = g$, and $D(f,g) = D(g,f)$. So the only issue is the triangle inequality. But if $f, g, h \in C(E, E')$ then

$$d'(f(x), h(x)) \leq d'(f(x), g(x)) + d'(g(x), h(x)).$$

Therefore

$$\begin{aligned} D(f,h) &\leq \max\{\, d'(f(x), g(x)) + d'(g(x), h(x)) : x \in E \,\} \\ &\leq \max\{\, d'(f(x), g(x)) : x \in E \,\} \\ &\qquad\qquad + \max\{\, d'(g(x), h(x)) : x \in E \,\} \\ &= D(f,g) + D(g,h). \end{aligned}$$

Thus D is a metric and $\big(C(E,E'), D\big)$ is a metric space. □

Break Time

Time for a break and some questions.

Convergence

Proposition

Suppose that E and E' are metric spaces with E compact. Let (f_n) be a sequence in $C(E,E')$. Then (f_n) converges to f in the metric space $\big(C(E,E'),D\big)$ if and only if (f_n) converges uniformly to f on E.

Proof.

This is just untangling definitions. Suppose that (f_n) converges to f in $C(E,E')$. Then given $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $D(f_n, f) < \epsilon$. But then for all $n \geq N$ we have

$$\max\{\, d'(f_n(x), f(x)) : x \in E \,\} < \epsilon.$$

That is, $n \geq N$ implies $d'(f_n(x), f(x)) < \epsilon$ for all $x \in E$. Therefore (f_n) converges uniformly to f on E.

Proof

Proof Continued.

Conversely, suppose that (f_n) converges uniformly to f on E. Then given $\epsilon > 0$, there is a $N \in \mathbf{N}$ such that $n \geq N$ implies that

$$d'(f_n(x), f(x)) < \epsilon \quad \text{for all } x \in E$$

Since the maximum of $d'(f_n(x), f(x))$ is attained on E, it follows that $D(f_n, f) < \epsilon$ if $n \geq N$. This shows that (f_n) converges to f in $C(E, E')$. $\square$

Completeness

Theorem

Suppose that E and E' are metric spaces with E compact and E' complete. Then $\big(C(E,E'),D\big)$ is complete.

Proof.

Let (f_n) be a Cauchy sequence in $C(E,E')$. Then given $\epsilon>0$, there is a $N\in\mathbf{N}$ such that $m,n\ge N$ implies $D(f_n,f_m)<\epsilon$. But then

$$\max\{\, d'(f_n(x),f_m(x)) : x\in E\,\}<\epsilon,$$

and

$$d'(f_n(x),f_m(x))<\epsilon \quad \text{for all } x\in E.$$

This shows that (f_n) is uniformly Cauchy. Since E' is complete, we know that there is a function $f:E\to E'$ such that (f_n) converges uniformly to f. Since the f_n are continuous, we know that f is continuous. Thus $f\in C(E,E')$ and (f_n) converges to f in $C(E,E')$. This completes the proof. □

Break Time

Time for a break and questions.

Calculus!

Remark

Having gone off the deep end—abstraction-wise—with $C(E, E')$, let's get real and consider the derivatives of real-valued functions on subsets of $\mathbf{R}$. Note that if $U \subset \mathbf{R}$ is open, then every $x_0 \in U$ is a cluster point of U and it makes sense to consider $\lim_{x \to x_0} g(x)$ for any function g defined on $U \cap \mathscr{C}\{x_0\}$. Since we've defined a neighborhood of x_0 to be a set containing an open set containing x_0, we can consider such limits for any g defined on a neighborhood of x_0.

Derivatives

Definition

Suppose f is a real-valued function defined on a neighborhood of $x_0 \in \mathbf{R}$. Then we say that f is differentiable at x_0 is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{$\dagger$}$$

exists. In that case, we write $f'(x_0)$ for the (unique) value of the limit and call $f'(x_0)$ the derivative of f at x_0.

Remark

It is easy to verify that we can replace ($\dagger$) with

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

for h is a neighborhood of 0.

Linear Approximation

Remark

If $f'(x_0)$ exists, then $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$. Alternatively, $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) = 0$. Equivalently,

$$\lim_{x\to x_0} \frac{f(x) - L(x)}{x - x_0} = 0.$$

where $L(x) = f(x_0) + f'(x_0)(x - x_0)$. This is a very strong way of saying that the linear function $L(x)$ is a good approximation to $f(x)$ near x_0. For example, if f is continuous at x_0 and $M(x) = M(x_0) + k(x - x_0)$ is any linear function, then

$$\lim_{x\to x_0} f(x) - M(x) = 0$$

if and only if $M(x_0) = f(x_0)$. Clearly this is unimpressive. (Look at some pictures!)

The Best Linear Approximation

Proposition

Suppose that f is a real-valued function defined on a neighborhood of $x_0 \in \mathbf{R}$ which is continuous at x_0. Suppose that $M(x) = M(x_0) + k(x - x_0)$ is a linear function such that

$$\lim_{x \to x_0} \frac{f(x) - M(x)}{x - x_0} = 0.$$

Then f is differentiable at x_0 and $M(x) = f(x_0) + f'(x_0)(x - x_0)$.

Proof.

Since f and M are both continuous at x_0, we have

$$\begin{aligned} f(x_0) - M(x_0) &= \lim_{x \to x_0} f(x) - M(x) \\ &= \lim_{x \to x_0} \frac{f(x) - M(x)}{x - x_0}(x - x_0) = 0. \end{aligned}$$

Therefore $M(x_0) = f(x_0)$.

Proof

Proof Continued.

Then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - M(x_0) - k(x - x_0) + k(x - x_0)}{x - x_0}$$

$$\lim_{x \to x_0} \left[\frac{f(x) - M(x)}{x - x_0} + k \right] = 0 + k = k.$$

Thus f is differentiable at x_0 and $k = f'(x_0)$. □

Better Than Continuous

Proposition

If f is differentiable at x_0, then f is continuous at x_0.

Proof.

Since f is defined in a neighborhood of x_0, it follows that f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$. This is equivalent to $\lim_{x \to x_0} \big[f(x) - f(x_0) \big] = 0$. But

$$\begin{aligned} \lim_{x \to x_0} \big[f(x) - f(x_0) \big] &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0. \end{aligned}$$

$\square$

Differentiable Functions

Definition

If U is an open set in $\mathbf{R}$ that we say that a real-valued function on U is differentiable if it is differentiable at every point of of U. Then we obtain a new function, $f' : U \to \mathbf{R}$, in the obvious way. The notation $\frac{df}{dx}(x)$ is also used.

Remark

While a differentiable function is necessarily continuous, a continuous function need not be differentiable. For example, consider $f(x) = |x|$. To see if $f'(0)$ exists, we consider the limit

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

where you can show does not exist. In fact, there are continuous functions on $\mathbf{R}$ that don't have a derivative at a single point! However, proving that such functions exist is beyond the scope of this course.

Rules To Differentiate With

The following are routine consequences of the definition.

Lemma

If $f(x) = c$ for all $x \in \mathbf{R}$, then $f'(x) = 0$ for all $x \in \mathbf{R}$. If $f(x) = x$ for all $x \in \mathbf{R}$, then $f'(x) = 1$ for all $x \in \mathbf{R}$.

Theorem

Suppose that f and g are differentiable at x_0. Then so are $f \pm g$, fg, and if $g(x_0) \neq 0$, $\frac{f}{g}$. Futhermore,

1. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$,
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$, *and*
3. $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$.

Proof.

(1) This is routine and you should convince yourself of that.

(2) We compute

$$\begin{aligned}
\lim_{x\to x_0} &\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}g(x)+ \\
&\qquad\qquad \lim_{x\to x_0} f(x_0)\frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0)g(x_0)+f(x_0)g'(x_0)
\end{aligned}$$

where we used the fact that g is continuous at x_0 to conclude that $\lim_{x\to x_0} g(x) = g(x_0)$.

Proof

Proof Continued.

(3) First we consider $\frac{1}{g(x)}$. Since $g(x_0) \neq 0$ and g is continuous at x_0, we can conclude that $g(x) \neq 0$ near x_0. Hence we can compute that

$$\lim_{x \to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x \to x_0} \frac{g(x_0) - g(x)}{(x - x_0)g(x)g(x_0)}$$
$$= -\frac{g'(x_0)}{g(x_0)^2}.$$

Now we can use the product rule (aka (2)) to compute

$$\Big(\frac{f}{g}\Big)'(x_0) = \Big(f\Big(\frac{1}{g}\Big)\Big)'(x_0) = f'(x_0)\frac{1}{g(x_0)} + f(x_0)\frac{-g'(x_0)}{g(x_0)^2}$$
$$= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

□

Enough

1. That is enough for today.