Math 63: Winter 2021
Lecture 17

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Getting Started

1. We should be recording.
2. Remember it is better for me if you have your video on so that I don't feel I'm just talking to myself.
3. Our midterm will be available Friday after class and due Sunday by 10pm. It will cover through Today's lecture and hence all of Chapter V.
4. You will have four hours for the exam plus the usual 30 minutes uploading time. Plan your window now.
5. Time for some questions!

Rules To Differentiate With

The following are routine consequences of the definition.

Lemma

If $f(x) = c$ for all $x \in \mathbf{R}$, then $f'(x) = 0$ for all $x \in \mathbf{R}$. If $f(x) = x$ for all $x \in \mathbf{R}$, then $f'(x) = 1$ for all $x \in \mathbf{R}$.

Theorem

Suppose that f and g are differentiable at x_0. Then so are $f \pm g$, fg, and if $g(x_0) \neq 0$, $\frac{f}{g}$. Futhermore,

1. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$,
2. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$, *and*
3. $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$.

More Formulas

Corollary

Suppose that f is differentiable at x_0 and $c \in \mathbf{R}$. Then cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

Corollary

Suppose that $n \in \mathbf{Z}$ and $f(x) = x^n$. Then $f'(x) = nx^{n-1}$.

Remark

We are being a bit sloppy here. It is implict that if $n < 0$ then f is not defined at 0, and if $n = 0$, we are just claiming that $f' = 0$.

Proof.

We have already dealt with the cases that $n = 0$ and $n = 1$. If $n \geq 1$, and we assume $\frac{d}{dx}(x^n) = nx^{n-1}$, then $\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x \cdot x^n) = x^n + x(nx^{n-1}) = (n+1)x^n$ so the result holds for all $n \geq 0$ by induction.

Proof

Proof Continued.

If $n < 0$, then $\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^{-n}}\right) = -\frac{-nx^{-n-1}}{x^{-2n}} = nx^{n-1}$. □

Proposition (The Chain Rule)

Suppose that U and V are open subsets of $\mathbf{R}$. Let $f : U \to V$ and $g : V \to \mathbf{R}$ be functions such that f is differentiable at $x_0 \in U$ and g is differentiable at $f(x_0) \in V$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof

Proof.

Let $A(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & \text{if } y \in V \text{ and } y \neq f(x_0), \text{ and} \\ g'(f(x_0)) & \text{if } y = f(x_0). \end{cases}$. Then $\lim_{y \to f(x_0)} A(y) = g'(f(x_0))$. It follows that A is continuous at $f(x_0)$. Therefore $\lim_{x \to x_0} A(f(x)) = A(f(x_0))$ since the composition of continuous functions is continuous. Therefore

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} A(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(f(x_0)) f'(x_0).\end{aligned}$$

□

Extreme Values

Proposition

Suppose that U is open in $\mathbf{R}$ and that $f : U \to \mathbf{R}$ has either a maximum or minimum value at $x_0 \in U$. Then either f is not differentiable at x_0 or $f'(x_0) = 0$.

Proof.

Assume that $f'(x_0)$ exists. Then

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \text{, and} \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

is continuous at x_0. Hence if $f'(x_0) > 0$, then is a $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $g(x) > \frac{f'(x_0)}{2} > 0$. But if $f(x_0)$ is either a maximum or a minimum, then $f(x) - f(x_0)$ is either always non-positive or always non-negative. Since $g(x) \neq 0$ if $0 < |x - x_0| < \delta$, it follows that $f(x) - f(x_0)$ is always either always positive or always negative. But $x - x_0$ can be both negative or positive. Hence $g(x)$ can't always be positive. Thus $f'(x_0) \leq 0$. But a similar argument implies $f'(x_0) \geq 0$. □

Rolle's Theorem

Lemma (Rolle's Theorem)

Suppose $a < b$ in $\mathbf{R}$, that $f : [a, b] \to \mathbf{R}$ is continuous and differentiable on (a, b). If $f(a) = f(b)$, then there is a $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

By the Extreme-Value Theorem, f attains its maximum and minimum on $[a, b]$. If both these values occur at the endpoints a and b, then f is constant and $f'(c) = 0$ for all $c \in (a, b)$. Otherwise, either the maximum or minimum must occur at some $c \in (a, b)$. By the previous proposition, $f'(c) = 0$. □

The Mean Value Theorem

Theorem (The Mean Value Theorem)

Suppose that $a < b$ *in* $\mathbf{R}$ *and that* $f : [a, b] \to \mathbf{R}$ *is continuous and differentiable on* (a, b). *Then there is a* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark

In my calculus courses, I emphasize that the MVT says that there is always a point where the instantaneous rate of change equals the average rate of change. (Picture)

Proof

Proof.

Let $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{(b-a)}(x-a)$. Then F is continuous on $[a,b]$ and differentiable on (a,b). Since $F(b) = F(a) = 0$, there is a $c \in (a,b)$ such that $F'(c) = 0$. The result follows from this. □

Corollary

Suppose that $f : (c,d) \to \mathbf{R}$ is differentiable and $f'(x) = 0$ for all $x \in (c,d)$. Then f is constant.

Proof.

It suffices to see that if $a, b \in (c,d)$, then $f(a) = f(b)$. But we may assume $a < b$, then the MVT implies
$f(b) - f(a) = \frac{f(b)-f(a)}{b-a}(b-a) = f'(c)(b-a) = 0$. □

Corollary

If f and g are differentiable real-valued functions on an open interval such that $f' = g'$, then f and g differ by a constant.

Increasing and Decreasing Functions

Definition

A real-valued function on a set $U \subset \mathbf{R}$ is called increasing (strictly increasing) if $a < b$ in U implies $f(a) \leq f(b)$ ($f(a) < f(b)$). On the other hand, f is called decreasing (strictly decreasing) if $a < b$ in U implies $f(a) \geq f(b)$ ($f(a) > f(b)$).

Proposition

Suppose that a real-valued function on an open interval is such that $f'(x) > 0$ for all x in the interval. Then f is strictly increasing on that interval.

Proof.

Suppose that $a < b$ in the interval. Then
$f(b) - f(a) = \frac{f(b)-f(a)}{b-a}(b - a) = f'(c)(b - a) > 0.$ □

Break Time

Time for a break and some questions.

Higher Order Derivatives

Remark

If f is a real-valued function which is differentiable on an open set $U \subset \mathbf{R}$, then we can consider the function $f' : U \to \mathbf{R}$. If f' is differentiable, then we say that f is twice differentiable and variously write f'' or $f^{(2)}$ for the second derivative $(f')'$. Of course, we can continue the madness and consider higher order derivatives $f''' = f^{(3)}$, and so on—if the derivatives exist. If f is n-times differentiable, we also write $\frac{d^n f}{dx^n}$ for $f^{(n)}$. To make some formulas work, we agree that $f^{(0)} = f$ (the 0^{th}-derivative).

Notation

We will also employ the factorial notation: $0! := 1$ and $(n+1)! = (n+1)n!$ for $n \geq 1$. Or less pedantically, $n! = 1 \cdot 2 \cdots n$.

A Lemma

It will be convenient to establish the following technical result.

Lemma

Suppose that $f : U \to \mathbf{R}$ is $(n+1)$-times differentiable. If $b \in U$, define a function on $R_n(b, \cdot) : U \to \mathbf{R}$ by

$$f(b) = f(x) + f'(x)(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(b-x)^n + R_n(b,x).$$

Then

$$\frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n.$$

Proof

Proof Continued.

Without the unusual structure, we have

$$R_n(b,x) = f(b) - f(x) - \sum_{j=1}^{n} f^{(j)}(x)\frac{(b-x)^j}{j!}.$$

Therefore $R_n(b,\cdot)$ is differentiable and using the product and chain rules, we have

$$\frac{d}{dx}R_n(b,x) = -f'(x) + \sum_{j=1}^{n}\Big[f^j(x)\frac{(b-x)^{j-1}}{(j-1)!} - f^{(j+1)}(x)\frac{(b-x)^j}{j!}\Big]$$

which telescopes to

$$= -f^{(n+1)}(x)\frac{(b-x)^n}{n!}.$$

□

Taylor's Theorem

Theorem (Taylor's Theorem)

Suppose that $U \subset \mathbf{R}$ is open and that $f : U \to \mathbf{R}$ is $(n+1)$-times differentiable on U. Then for any $a, b \in U$, we have

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots$$
$$\cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

where c lies between a and b. (In the degenerate case $a = b$, we can let $c = a$.)

Remark

In the case $n = 0$, this reduces to a milder form of the Mean Value Theorem.

Proof

Proof.

If $a = b$, the result is trivial. The awkward statement of the last lemma is so that we can see that we just need to show that, in the notation of that lemma,

$$R_n(a, b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c between a and b. Since $a \neq b$, there is a real number K such that

$$R_n(a, b) = K\frac{(b-a)^{n+1}}{(n+1)!}.$$

Proof

Proof Continued.

Define $\varphi : U \to \mathbf{R}$ by

$$\varphi(x) = R_n(b, x) - K\frac{(b-x)^{n+1}}{(n+1)!}.$$

Clearly φ is differentiable on U and $\varphi(a) = \varphi(b) = 0$. Therefore φ restricted to the interval $[a, b]$ if $a < b$ or $[b, a]$ if $b < a$ satisfies the hypotheses of Rolle's Theorem. Thus there is a c between a and b such that $\varphi'(c) = 0$. But by our lemma,

$$\varphi'(x) = -\frac{f^{(n+1)}(x)}{n!}(b-a)^n + K\frac{(b-x)^n}{n!}.$$

Thus $K = f^{(n+1)}(c)$ and $R_n(a, b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$ as claimed. □

An Interpretation

Remark

Suppose that $f : U \to \mathbf{R}$ is $(n+1)$-times differentiable on an open set U. If $a \in U$, let

$$P_n(a,x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We call $P_n(a,x)$ the n^{th}-degree Taylor polynomial for f about $x = a$. Note that $P_1(a,x)$ is just the linear approximation to $f(x)$ near a given by the derivative. That is $P_1(a,x)$ is the unique polynomial such that $P_1(a,a) = f(a)$ and $P_1'(a,a) = f'(a)$. It is not so hard to check that $P_n(a,x)$ is the unique polynomial of degree at most n such that $P_n^{(k)}(a,a) = f^{(k)}(a)$ for all $0 \le k \le n$. Then $f(x) = P_n(a,x) + R_n(a,x)$ and $R_n(a,x)$ is just the difference $f(x) - P_n(a,x)$. Thus $|R_n(a,x)|$ is just the error in approximating $f(x)$ with $P_n(a,x)$.

Example

Example

Let $f(x) = \frac{1}{x}$ and let $a = 1$. Then $f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$. Thus $\frac{f^{(j)}(1)}{j!} = (-1)^j$ and the Taylor polynomial of degree n about $x = 1$ is

$$P_n(1,x) = 1 - (x-1) + (x-1)^2 - \cdots + (-1)^n(x-1)^n.$$

It is fun to use `Mathematica` to draw some pictures. Note that if $x \in [1,b]$, then

$$|R_n(1,x)| = \left|\frac{(x-1)^{n+1}}{c^{n+2}}\right| \le |b-1|^{n+1}$$

since $c \in (1,b)$.

Enough

1. That is enough for today.