Math 63: Winter 2021
Lecture 25

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Getting Started

1. We should be recording.
2. The final will be administered in a manner similar to the prelim and midterm exams and will be available from Saturday, March 13, at 8am EST, to Monday, March 15 at 10pm EST.
3. Time for some questions!

Review

Theorem

Suppose that $\sum_{n=0}^{\infty} c_n(x-a)^n$ *is a power series. Then exactly one the following three cases applies.*

1. *The series converges absolutely for all* $x \in \mathbf{R}$.
2. *There is a* $R > 0$ *such that the series converges absolutely if* $|x-a| < R$ *and diverges if* $|x-a| > R$.
3. *The series converges only if* $x = a$.

Furthermore, if R_1 *is such that* $0 < R_1 < R$ *in case (2) or for any* $R' > 0$ *in case (1), the convergence is uniform on* $\{\, x : |x-a| \leq R_1 \,\}$.

Review

Theorem

Suppose that $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R \in (0, \infty]$ *and we define*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{for all } x \in (a-R, a+R).$$

Then f *is differentiable on* $(a-R, a+R)$ *and for all* $x \in (a-R, a+R)$

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \quad \text{and} \quad \int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}.$$

Remark

It should be clear that if $R = \infty$, then $(a-R, a+R)$ is meant to be interpreted as $(-\infty, \infty) = \mathbf{R}$.

Review

Corollary

Suppose that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*. (That is,* $R \in (0, \infty]$*.) Then* f *has derivatives of all orders on* $(a-R, a+R)$ *and for all* $n \geq 0$

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

In particular, the power series expansion for f *about* a *is unique.*

Question

Now suppose that f has derivatives of all orders on an interval $U \subset \mathbf{R}$ with $a \in U$. Then we can form the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots .$$

Let's suppose the series has a positive radius of convergence R and defines a function g on $(a-R, a+R)$. The obvious question is whether $g = f$!.

Question (Question Continued)

But we know that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x,a)$$

where

$$R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

with c between x and a. Therefore we recover $f(x)$ exactly when $\lim_n R_n(x,a) = 0$.

Example

Example

Let $f(x) = \log(1+x)$. Then $f'(x) = \frac{1}{1+x}$, $f''(x) = -\frac{2}{(1+x)^2}$, $\dots$.
In general,

$$f^{(n)}(x) = -(1)^{n-1}\frac{(n-1)!}{(1+x)^n} \quad \text{for } n \geq 1 \text{ and } x > -1.$$

Thus $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ and the series for f about $x = 0$ must be of the form

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

We want to know when this converges to $\log(1+x)$. Note that

$$R_n(x,0) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{1}{n+1}\Big(\frac{x}{1+c}\Big)^{n+1}$$

for some c between 0 and x.

Example (Continued)

If $x \geq 0$, then we have

$$|R_n(x,0)| \leq \frac{1}{n+1} x^{n+1}.$$

Then $\lim_n |R_n(x,0)| = 0$ if $0 \leq x < 1$. On the other hand, if $-1 < x \leq 0$, then

$$|R_n(x,0)| \leq \frac{1}{n+1}\Big(\frac{|x|}{1-|x|}\Big)^{n+1}.$$

Then we can show that $\lim_n |R_n(x,0)| = 0$ only when $-\frac{1}{2} < x \leq 0$.

This shows that the series converges to $f(x) = \log(1+x)$ only when $x \in (-\frac{1}{2}, 1)$. This seems wrong since the series converges if $|x| < 1$.

Example

Example (Improved)

If $|x| < 1$, then we can sum the a geometric series to get

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + \cdots .$$

Then we can integrate term-by-term to get

$$\log(1+x) = \int_0^x \frac{1}{1+t}\,dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ,$$

which is valid for all $|x| < 1$.

This is obviously superior to repeated differentiation and estimating the error term.

Break Time

Time for a break and some questions.

Another Question

Question

Before the break, we established that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for all } |x| < 1.$$

But both sides make sense if $x = 1$. Is it the case that

$$\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ?$$

As we shall show, the answer is “yes”. But it should be observed that the convergence is very slow—remember the error $|s_n - \log(2)|$ is only bounded by $\frac{1}{n+1}$ which doesn’t get small very fast. But it is a pretty formula.

Abel's Theorem

Theorem (Abel's Theorem)

Suppose that $\sum_{n=1}^{\infty} a_n$ *is a convergent series of real numbers. Then*

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

is uniformly convergent on $[0,1]$ *and* f *is continuous on* $[0,1]$*. In particular,*

$$\sum_{n=1}^{\infty} a_n = f(1) = \lim_{x \nearrow 1} f(x).$$

Remark

This answers our question. In our case $f(x)$ and $\log(1+x)$ are equal on $[0,1)$. Since both are continuous, they must be equal at $x = 1$ as well.

Proof

Proof.

It suffices to see that $\sum_{n=1}^{\infty} a_n x^n$ is uniformly convergent on $[0,1]$. Then the sum is clearly continuous and the rest is routine.

Given $\epsilon > 0$, there is a $N \in \mathbf{N}$ such that $m > n \geq N$ implies $|a_{n+1} + \cdots + a_m| < \epsilon$. Using our Cauchy Criterion, it will suffice to prove that

$$\Big| \sum_{k=n+1}^{m} a_k x^k \Big| < \epsilon \quad \text{for all } x \in [0,1].$$

Note that the sequence $(a_k)_{k=n+1}^{\infty}$ has partial sums $s_r = a_{n+1} + \cdots + a_{n+r}$ bounded by ϵ.

Proof Continued.

By our Summation by Parts formula,

$$\left|\sum_{k=n+1}^{m} a_k x^k\right| = \left|s_{m-n}x^m + \sum_{k=n+1}^{m-1} s_{k-n}(x^k - x^{k+1})\right|.$$

Since $0 \leq x \leq 1$, $x^k \geq x^{k+1}$, we have

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k x^k\right| &\leq |s_{m-n}|x^m + \sum_{k=n+1}^{m-1} |s_{k-n}|(x^k - x^{k+1}) \\ &\leq \epsilon \cdot x^m + \epsilon \cdot (x^{n+1} - x^m) = \epsilon x^{n+1} \leq \epsilon.\end{aligned}$$

This completes the proof. □

Break Time

Time for a break and some questions.

Another Example

Example (Cheating Again)

We can define a smooth function $F : \mathbf{R} \to \mathbf{R}$ by

$$F(x) = \int_0^x \frac{1}{1+t^2}\,dt.$$

To feel impressed with ourselves, we can remember that $F(x) = \arctan(x)$, but we won't use any properties of arctan until the very end and then only to show off a bit. We'd like to express arctan as the sum of a power series about $x = 0$. Repeated differentiation is not going to be helpful let alone trying to estimate $R_n(x, 0)$. But we are far from lost.

Example

Example (Continued)

Since $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for all $|x| < 1$, we also have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \quad \text{for all } |x| < 1.$$

Therefore we can integrate term-by-term to get

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for all } |x| < 1.$$

Question

If $F(x) = \arctan(x)$, what is $F^{(2020)}(0)$? How about $F^{(2021)}(0)$?

Answer

Remark (The Answer)

Since we have $F(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < 1$, we must have

$$c_n = \frac{F^{(n)}(0)}{n!}.$$

Since $c_{2k} = 0$ for $k \geq 0$, we must have $F^{(2020)}(0) = 0$. However, $c_{2k+1} = (-1)^k \frac{1}{2k+1}$. Since $2021 = 2(1010) + 1$,

$$F^{(2021)}(0) = (2021)! \cdot \frac{(-1)^{1010}}{2021} = (2020)!.$$

Remark

We can also use Abel's Theorem to conclude that

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Some Oddities

Remark

Now I want to display some extreme examples. I will just sketch the proofs since filling in all the details would be painful and the point is just to be aware of what's out there.

Lemma

For all $n = \{0, 1, 2, \dots\}$, *we have*

$$\lim_{x \to 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = 0.$$

Remark

In the good old days, we would argue that this is equivalent to the assertion that $\lim_{x\to\infty} x^n e^{-x^2} = 0$ and use L'Hospital's rule, but I'd rather sketch a "Math 63 Proof".

Proof

Sketch of the Proof.

I leave it to you to show that $\log(x) \le x$ for all $x > 0$. Then

$$0 \le \left| \frac{1}{x^n} e^{-\frac{1}{x^2}} \right| = \exp\Big[-\frac{1}{x^2} + n\log\big(\frac{1}{x}\big)\Big] \le \exp\Big[-\frac{1}{x}\big(\frac{1}{x} - n\big)\Big].$$

But by considering $\lim_{x\nearrow 0}$ and $\lim_{x\searrow 0}$ separately, it is not hard to see that

$$\lim_{x\to 0} \exp\Big[-\frac{1}{x}\big(\frac{1}{x} - n\big)\Big] = 0.$$

This suffices. □

A Function

Proposition (Chapter VI #26)

Define $f : \mathbf{R} \to \mathbf{R}$ *by*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0. \end{cases}$$

Then f *has derivatives of all orders at* $x = 0$ *and* $f^{(n)}(0) = 0$ *for all* $n = \{0, 1, 2, \dots\}$.

Remark

Note that by the chain rule, f is obviously smooth on $\mathbf{R} \setminus \{0\}$. Only $x = 0$ is interesting.

Proof

Sketch of the Proof.

Note that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} e^{-\frac{1}{x^2}} = 0$$

by our limit lemma. Now the game is proceed by induction. For this we need another lemma.

A Lemma

Lemma

If $n \in \mathbf{N}$ and $x \neq 0$, then $f^{(n)}(x) = P(\frac{1}{x})e^{-\frac{1}{x^2}}$ were $P \in \mathbf{R}[x]$ is a polynomial.

Proof of the Lemma.

This is easy if $n = 1$. On the other hand, if $f^{(n)}(x) = P(\frac{1}{x})e^{-\frac{1}{x^2}}$, then

$$\begin{aligned} f^{(n+1)}(x) &= P'(\tfrac{1}{x})(\tfrac{-1}{x^2})e^{-\frac{1}{x^2}} + P(\tfrac{1}{x})(\tfrac{2}{x^3})e^{-\frac{1}{x^2}} \\ &= \underbrace{\Big[P'(\tfrac{1}{x})(\tfrac{-1}{x^2}) + P(\tfrac{1}{x})(\tfrac{2}{x^3})\Big]}_{Q(\frac{1}{x})} e^{-\frac{1}{x^2}} \end{aligned}$$

where Q is a polynomial. □

Proof

Proof of the Proposition Continued.

In view of the lemma, we can assume that

$$f^{(n)}(x) = \begin{cases} P(\frac{1}{x})e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \text{ and} \\ 0 & \text{if } x = 0 \end{cases}$$

for some polynomial P. But then

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(0)}(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{1}{x} \cdot P(\frac{1}{x})e^{-\frac{1}{x^2}} \end{aligned}$$

which is equal to zero since $\lim_{x \to 0} \frac{1}{x^n} e^{-\frac{1}{x^2}} = 0$ for all n. □

Example

Let f be our smooth example where $f^{(n)}(0) = 0$ for all $n \geq 0$. If we could expand f into a power series about $x = 0$, we would have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$$

Thus the only possible power series expression for f converges to f only at $x = 0$.

Just so that You Know

Example

Let's suppose that we believe that

$$f(x) = \int_0^\infty e^{-t}\cos(t^2x)\,dt = \lim_{n\to\infty}\int_0^n e^{-t}\cos(t^2x)\,dt$$

defines a smooth function and that the derivative can be computed by "differentiating" under the integral sign. For example,

$$f'(x) = -\int_0^\infty e^{-t}t^2\sin(t^2x)\,dx.$$

Give that, using integration by parts, it is not so hard to verify that

$$f^{(2n)}(0) = \int_0^\infty e^{-t}t^{4n}\,dt = \pm(4n)!.$$

But then the $(2n)^{\text{th}}$ term of the power series expansion for f is $\pm\frac{(4n)!}{(2n)!}x^{2n}$ and these terms tends to zero with n only when $x = 0$. Therefore the power series expansion for this f converges only at $x = 0$.

Enough

1. That is enough for today.