# Potential topics for the project

MATH 68

October 10, 2019

## 1 Related to posets

**Random walks on the faces of a hyperplane arrangement** This area of research has been very popular in the '90's and gave birth to a literature that is still cited today. Some people from Dartmouth contributed to this area, including Philip Hanlon and Daniel Rockmore.

The idea is to define some random walks between regions of hyperplane arrangements, such that the probability of transitions are given by the arrangement itself. Among these walks, there are a lot of card shuffles.

One good reference (there are many) : [BHR99].

**Promotion and evacuation for posets** A *linear extension* is a class of permutations that agree with a poset. Promotion and evacuation are operations that we apply to linear extensions, as well as tableaux. I would like you to explain the interplay between these structures. References: [EC1, §3.20], [Ful97].

## 2 Related to Pólya theory

**The Gessel-Reutenauter bijection** This bijection links the number of necklaces (we will see next week what it is) and the number of permutations with a given cycle type. The reference for that is this highly influential paper [GR93].

## 3 Related to tableaux

**Representations of $GL_n$ : Schur functions** Symmetric functions are multivariable functions who are stable under the permutation of the variables. Among the basis for the symmetric functions, the Schur functions have a very special meaning, since they also are the characters of the general linear group, $GL_n$. A presentation reflecting that would be great. Again, many resources exist: [Mac15], [Sag01, §4.4, 4.6] or [EC2, §7.10, 7.15], among others.

**The hook-length formula** The hook-length formula counts the number of standard Young tableaux of a given shape. This is given by the *hook-length* of every cell (we will see that later). It has been discovered in 1953 by J. S. Frame, G. de B. Robinson and R. M. Thrall, but their proof is not very insightful. Many other people gave other proofs, and it would be great to hear some main ideas behind them. There is a bijective proof of this formula [NPS97].

References: You can choose the original paper, a paper that gives an alternative proof (more elegant, probabilistic, etc.), or a textbook (for example, [Sag01, §3.10] [Ful97, §4.3] or [EC2, §7.21]).

**Viennot's construction of the RSK correspondence** We will soon see the RSK correspondence between pairs of standard tableaux and permutations. We will see it from an algorithmic point of view, but another one is also very interesting: Xavier Viennot gave a geometric description of it, and it is beautiful! Reference: [Sag01, §3.6]

**Flag varieties, through tableaux** Tableaux have a lot of applications in geometry, and there is a lot of research around this interaction. More specifically, the grassmanian $G(n, k)$ is the set of all $k$-dimensional subspaces in a vector space of dimension $n$, and the Schubert Calculus is interested in counting the intersections in a grassmanian. It is highly enumerative and can be explained through tableaux. Possible reference: [Ful97, Chapter 9]

**Jacobi-Trudi determinants - Alex**

# 4 Other projects

**Cyclic sieving phenomenon** Take your favorite combinatorial object and its generating function. Your object might be an occurrence of the *cyclic sieving phenomenon*. This happens when the roots of unity of the generating function count the number of symmetry classes of an object. There are a lot of occurrences of the CSP!

Resources: A short paper in the notices of the AMS [RSW14], or a journal paper [RSW04].

**The Matrix-tree theorem - Ben**

**Cluster algebras** This is a very trendy area of current research, in the field of combinatorial algebra (which, of course, has some relations with algebraic combinatorics.) This has connections with representation theory, geometry and Catalan numbers.

Portal on Cluster Algebras (many resources!) The first chapters of a forthcoming textbook on the subject, written by the mathematicians who created that field (Sergei Fomin and Andrei Zelevinsky), along with Lauren K. Williams.

**Various projects that I do not describe**

- The plactic monoid on semistandard Young tableaux
- Crystals of tableaux and their meaning
- Multiplication of tableaux
- Group actions on Boolean algebra
- Or suggest your own project

## Software projects

**Computing the characters with the Murnaghan-Nakayama rule** The Murnaghan-Nakayama rule is a powerful way to compute the characters of the symmetric group from a Ferrers diagram, which is a hard task otherwise. We will see the theory behind it in class, but if you want to implement it, I can explain it to you before that lecture.

**Identify a poset isomorphic to a given poset** I would like to have a software to which I can give a poset, for example by describing cover relations, and that would return candidates for isomorphic posets. If I have the candidates, then it is easy to check that their Hasse diagrams are equal. Findstat (`http://www.findstat.org/`) is a very elaborated project, but the idea is similar.

**Write a tutorial on how to use a library of functions** If you want to explore a library of functions for doing algebraic combinatorics in SageMath [Sage] (or another software), you can then write a tutorial on how to use it. That would count as your project.

## References


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