Math 68 - Algebraic Combinatorics
Hyperplane Arrangements with posets

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Goal of the lecture: Answer the question "Into how many connected components a space of dimension n is split if there are m hyperplanes splitting it?" The answer, of corse, depends on how these hyperplanes are placed.

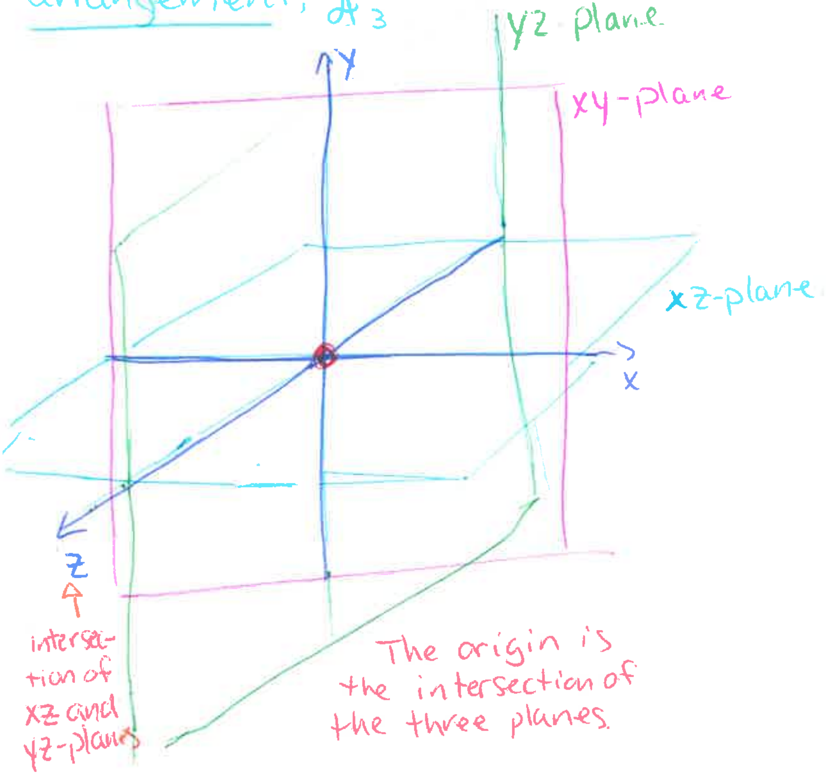
An (affine) hyperplane is a translate of a space of codimension 1 (i.e. a linear hyperplane).

or dimension $n-1$, where n is the dimension of the space.

A hyperplane arrangement is a finite set of hyperplanes in the same vector space.

Examples

The coordinates hyperplanes, in $\mathbb{R}^3$, form the boolean arrangement, $\mathcal{A}_3$



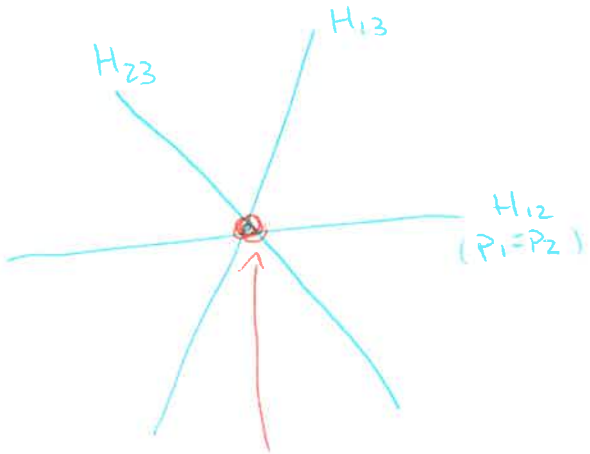
intersection of xz and yz-planes

The origin is the intersection of the three planes.

The braid arrangement

Hyperplanes $H_{ij} = \{ p=(p_1, \ldots, p_n) \mid p_i = p_j \}$ # of hyperplanes: $\binom{n}{2}$

$\mathcal{B}_3$, the braid arrangement in $\mathbb{R}^3$, admits the following projection onto a 2-dimensional space.



$H_{12} \cap H_{13} \cap H_{23}$ is the line $\langle (1,1,1) \rangle$

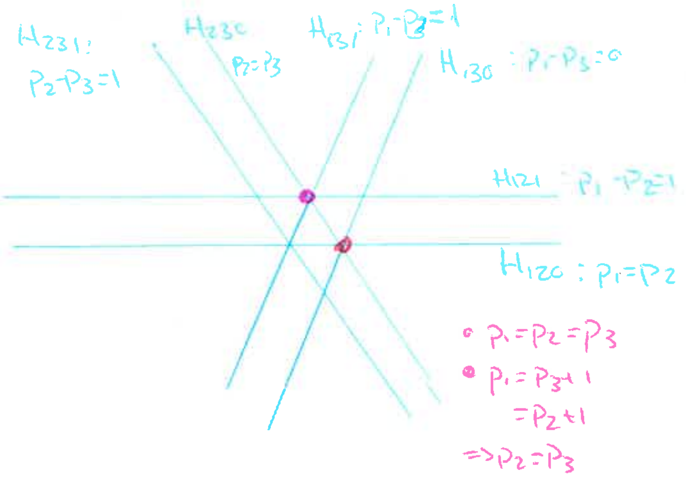
The Shi arrangement

This one is similar to the braid arrangement, but has twice as many hyperplanes:

$H_{ij0} = \{ p \mid p_i = p_j \}$

$H_{ij1} = \{ p \mid p_i - p_j = 1, i<j \}$

The 2-dimensional representation of the 3-dimensional arrangement is



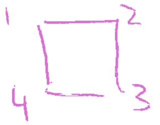
∘ $p_1 = p_2 = p_3$

● $p_1 = p_3 + 1$
$= p_2 + 1$
$\Rightarrow p_2 = p_3$

Arrangement associated to a graph

If $G = (V, E)$ has n vertices, this represents a subset of $\mathcal{B}_n$, the braid arrangement

The hyperplane H_{ij} belongs to the arrangement if $(i,j) \in E$.

Example: 

is a 4-dimensional arrangement with hyperplanes

$H_{12} = \{ p \mid p_1 = p_2 \}$

$H_{14} = \{ p \mid p_1 = p_4 \}$

$H_{23} = \{ p \mid p_2 = p_3 \}$

$H_{34} = \{ p \mid p_3 = p_4 \}$

Let $\mathcal{A}$ be an arrangement in a vector space of dimension n.

- The dimension of $\mathcal{A}$ is n (i.e the dimension of the space).
- The rank of $\mathcal{A}$ is the dimension of the space spanned by the normal vectors to the hyperplanes.

Examples

- The boolean arrangement $\mathcal{A}_n$ has both dimension and rank equal to n.
- The braid arrangement $\mathcal{B}_n$ has dimension n, but rank $n-1$. This is why we can easily draw $\mathcal{B}_3$ on paper
- The Shi arrangement is like the braid arrangement

- $\mathcal{A}$ is essential if rank $(\mathcal{A})$ = dim $(\mathcal{A})$, or, equivalently, if the normals to the hyperplanes span the whole space.
  - The boolean arrangement is essential.

- $\mathcal{A}$ is central if the intersection of all the hyperplanes is non-empty. Example: The braid arrangement is central, but not the Shi arrangement.

Intersection poset

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$.

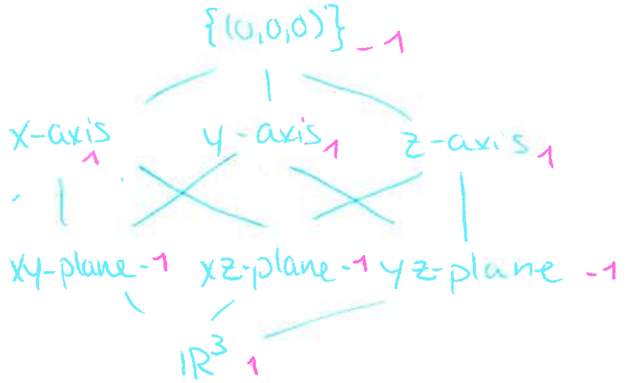
Let $L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including $\mathbb{R}^n$ as the intersection over the empty set.

Define $s \leq t$ in $L(\mathcal{A})$ if $s \supseteq t$ in $\mathcal{A}$ (reverse inclution)

Then, $(L(\mathcal{A}), \leq)$ is a poset with a $\hat{0}$ $(\mathbb{R}^n)$.

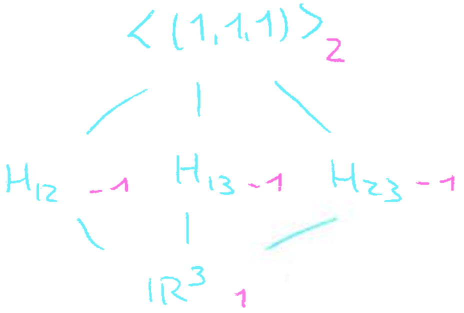
Examples

For the boolean arrangement $\mathcal{B}_3$:



The lattice is isomorphic to the boolean lattice.

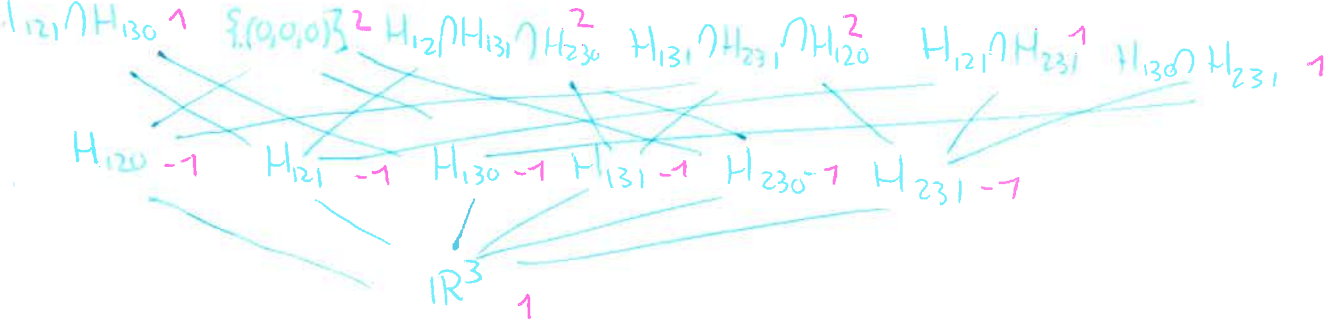
For the braid arrangement, $\mathcal{B}_3$.



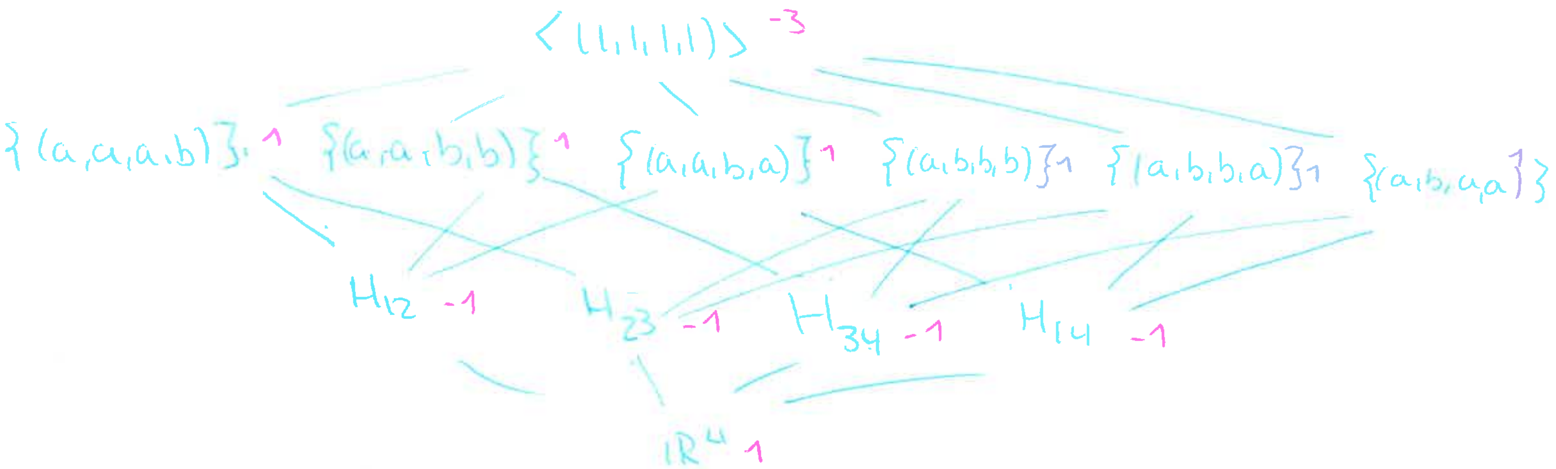
The lattice is isomorphic to the partition lattice Π_3.

For the Shi arrangement (in dimension 3)

Question: What is it isomorphic to?



For the arrangement associated to the graph  , with $a,b \in \mathbb{R}$



If $\mathcal{A}$ is central, $L(\mathcal{A})$ is a lattice (otherwise, $L(\mathcal{A})$ is called a meet-semi-lattice, i.e a poset such that the meet of two elements is always defined).

Regions

A region in a hyperplane arrangement is a connected component of $\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$.

Example

* There are 2^n regions for the boolean arrangement.
  In dimensions 2 and 3, we call them quadrant and octant. All the regions are unbounded.
* The braid arrangement $\mathcal{B}_3$ has six regions, all unbounded
* The Shi arrangement in $\mathbb{R}^3$ has 12 unbounded regions, and 4 bounded ones

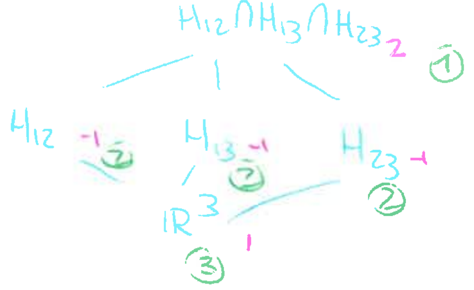
Definition

The characteristic polynomial $\chi_{\mathcal{A}}(x)$ of an arrangement is

$$\chi_{\mathcal{A}}(x) = \sum_{t\in L(\mathcal{A})} \mu(t)\, x^{\dim(t)}$$

Example

For $\mathcal{A} = \mathcal{B}_3$, $L(\mathcal{A})$ looks like



Möbius function

● dimension

The characteristic polynomial is $x^3 - 3x^2 + 2x$

Example

Let $\mathcal{A}$ be the arrangement associated to $\begin{smallmatrix}1 & \square & 2\\ 4 & & 3\end{smallmatrix}$.

Then,

$$\chi_{\mathcal{A}}(x) = x^4 - 4x^3 + 6x^2 - 3x.$$

Example

$$\chi_{\mathcal{A}_3}(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3.$$

Theorem (Zaslavsky, 1975)

Let $\mathcal{A}$ be an n-dimensional real vector space

Then,

(i) The number of regions is $(-1)^n \chi_{\mathcal{A}}(-1) = \sum_{u \in L(\mathcal{A})} |\mu(u)|$.

(ii) The number of relatively banded regions* is $(-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$

*see next page

Example

Using the characteristic polynomial computed earlier, we get that

- $\mathcal{B}_3$ has six regions, all unbanded.
- $\mathcal{A}_3$ has eight regions all unbanded.
- The Shi arrangement has characteristic polynomial $x^3 - 6x^2 + 9x$.

Therefore, it has 16 regions, including 4 unbounded regions, which corresponds to what we saw on the picture.

Question: How many regions does the Shi arrangement of dimension 4 have? (Hint: you do not have to draw it).

Hyperplanes in general position

Proposition (Schläfli, written 1850-1852, published 1901)

Let $\mathcal{A}$ be an arrangement of m hyperplanes in general position in $\mathbb{R}^n$. Then,

$$\chi_{\mathcal{A}}(x) = x^n - mx^{n-1} + \binom{m}{2} x^{n-2} - \dots + (-1)^n \binom{m}{n}.$$

In particular,

- * the number of regions is $\sum_{k=0}^{n} \binom{m}{k}$.
- * the number of bounded regions is $\sum_{k=0}^{n} (-1)^k \binom{m}{k} = \binom{m-1}{n}$.

The idea of the proof is to use a boolean arrangement with fewer hyperplanes.

Reference: Richard P. STANLEY. Enumerative Combinatorics, volume 1. § 3.11.

* A relatively bounded region is a region who is bounded in the vector space spanned by the normals to the hyperplanes.