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We are still trying to find the number of colorings...

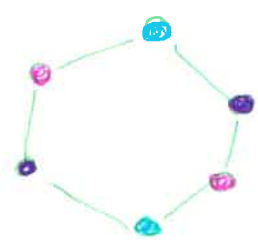
Problem: How many inequivalent colorings of the hexagon are there, under cyclic symmetries?

Claim: This is the number of orbits of G*. The reason for this is that two colorings are equivalent if and only if they are in the same orbit.

* acting on the colorings of the hexagon

The group acting on the hexagon is $\mathbb{Z}_6$, the cyclic group. Using Burnside's Lemma, its number of orbits is

$$\frac{1}{6}\sum_{\sigma\in\mathbb{Z}_6} \#\mathrm{fix}(\sigma).$$



Here, $\mathrm{fix}(\sigma)$ is the set of colorings fixed by σ.
Let $\pi = (123456)$, the rotation of $1/6$ of a turn. Then, the elements of $\mathbb{Z}_6$ are

σ	π	π^2	π^3	π^4	π^{-1}	$\pi^6 = e$
$\#\mathrm{Fix}(\sigma)$	n	n^2	n^3	n^2	n^1	n^6

Ex. of colorings

Thus, the number of orbits is

$$\frac{1}{6}\left(n^6+n^3+2n^2+2n\right)$$ by Burnside's Lemma,

and so is the number of different colorings.

Theorem

Let G be a group of permutations of a finite set X. Then, the number $N_G(n)$ of inequivalent (w.r.t G) colorings of X with n colors is

$$N_G(n) = \frac{1}{\#G}\sum_{\sigma\in G} n^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles of σ.

Proof

Let σ_n denote the action on the set of n-colorings, and σ the action on X.

We want to apply Burnside's Lemma, with σ_n (so we get the number of orbits of colorings, that is the number of inequivalent colorings)

For a coloring $f: X \to C$,

$$\sigma_n \cdot f(x) = f(\sigma(x)),$$

because σ is a group action.

And if f is fixed by σ_n, $f(x) = \sigma_n \cdot f(x) = f(\sigma(x))$, so $f \in Fix(\sigma_n)$ iff $f(x) = f(\sigma(x))$. We can apply this multiple times, so $f \in Fix(\sigma_n)$ iff $f(x) = f(\sigma^k(x))$, $\forall k \geq 1$

Fixing x, what are the elements of the form $\sigma^k(x)$? Just the elements in the cycle containing x. So all the elements in a cycle have to be the same color.

How many colorings can we choose, so they are fixed by σ? For each cycle, we have n choices, and all the elements in the cycle are of the same color, so $n^{c(\sigma)}$. (Example with number of cycles in $\mathbb{Z}_6$)

Back to the original question, we use Burnside's lemma to count the number of inequivalent colorings:

$$\frac{1}{\#G} \sum_{\sigma \in G} fix(\sigma_n) = \frac{1}{\#G} \sum_{\sigma \in G} n^{c(\sigma)}.$$

Number of colorings with a fixed number of each color

What if we fix the number of occurrences of each color?

Definition

The cycle type of the permutation $\sigma \in S_n$ is the tuple containing as a i-th item the number of cycles of length i in σ.*

$$type(\sigma) = (c_1, c_2, \dots, c_n)$$

*This will be the notation for this chapter, but it might change slightly next week.

Example

The cycle type of $34721568 = (13765)(24)(8)$ is

$$\underbrace{(1,1,0,0,\overset{i=5}{1},0,0,0)}_{\text{length } n}$$

Remark: $\sum_{1\le i\le n} i c_i = n$.

The cycle indicator is $Z_\sigma = z_1^{c_1} \cdots z_n^{c_n}$ and, for a group G, the cycle indicator of G is $\frac{1}{\#G}\sum_{\sigma\in G} Z_\sigma = \frac{1}{\#G}\sum_{\sigma\in G} z_1^{c_1}\cdots z_n^{c_n}$.

Example

1	2
3	4

Back to the example with the square, and consider these two groups:

(R) the rotations of the square. Its elements are $(1)(2)(3)(4)$, (1243), $(14)(23)$ and (1342). Then,

$$Z_R = \frac{1}{4}\left(z_1^4 + z_2^2 + 2z_4\right)$$

(D_1) the dihedral group contains R and the elements $(12)(34)$, $(13)(24)$, $(1)(23)(4)$ and $(14)(2)(3)$. Then,

$$Z_{D_1} = \frac{1}{8}\left(z_1^4 + 3z_2^2 + 2z_1^2 z_2 + 2z_4\right)$$

Theorem (Pólya's theorem, 1937).

Let G be a group of permutations of the n-element set X. Let $C = \{b_1, b_2, \dots\}$ the set of colors.

Let $K(i_1, i_2, \dots)$ be the set of inequivalent colorings with b_j used i_j times.

Then, the generating function of K is

$$F_G(r_1, r_2, \dots) = Z_G(r_1 + r_2 + \dots,\ r_1^2 + r_2^2 + \dots,\ r_1^3 + r_2^3 + \dots,\ \dots,\ r_1^{|G|} + r_2^{|G|} + \dots)$$

(where $Z_G = Z_G(z_1, z_2, z_3, \dots)$).

Example

Let G be the dihedral group and consider the inequivalent colorings of $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ with the colors r = red, p = purple, b = blue and g = green. Then

$$Z_{D_1}(r,p,b,g) = \frac{1}{8}\Big((r+p+b+g)^4 + 3(r^2+p^2+b^2+g^2)^2 + 2(r+p+b+g)^2(r^2+p^2+b^2+g^2) + 2(r^4+p^4+b^4+g^4)\Big)$$

expansion

$$\rightarrow = \frac{1}{8}\Big((1+3\cdot1+2\cdot1+2)(r^4+p^4+b^4+g^4)$$
$$+ (4+2\cdot2)(r^3p+r^3b+r^3g+p^3r+p^3b+p^3g+b^3r+b^3p+b^3g+g^3r+g^3b+g^3p)$$
$$+ (6+3\cdot2+2\cdot2)(r^2p^2+r^2b^2+r^2g^2+p^2b^2+p^2g^2+b^2g^2)$$
$$+ (12+2\cdot2)(r^2pb+r^2pg+r^2bg+p^2rb+p^2rg+p^2bg+b^2rp+b^2rg+b^2pg+g^2rp+g^2rb+g^2pb)$$
$$+ (24)\, rpbg\Big)$$

$$= (r^4+p^4+b^4+g^4) + (r^3p+r^3b+r^3g+p^3r+p^3b+p^3g+b^3r+b^3p+b^3g+g^3r+g^3b+g^3p)$$
$$+ 2(r^2p^2+r^2b^2+r^2g^2+p^2b^2+p^2g^2+b^2g^2)$$
$$+ 2(r^2pb+r^2pg+r^2bg+p^2rb+p^2rg+p^2bg+b^2rp+b^2rg+b^2pg+g^2rp+g^2rb+g^2pb)$$
$$+ 3\, rpbg$$

Evaluating Z_{D_1} in $(1,1,1,1)$, we get the total number of colorings.

$$Z_{D_1}(1,1,1,1) = 1(4) + 1(12) + 2(6) + 2(12) + 3$$
$$= 55,$$

which is compatible with what we found last week:

$$f_{D_1}(1,1,1,1) = \frac{1}{8}(4^4 + 2\cdot4^3 + 3\,4^2 + 2\,4) = 55.$$

Reference: Richard P. Stanley, *Algebraic Combinatorics* §7.