Math 68 - Algebraic Combinatorics

Tableaux

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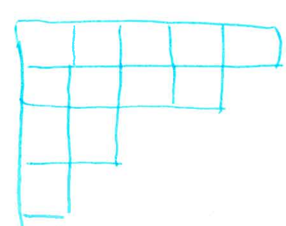
Motivation: Young tableaux (and variations on that theme) allow us to study in details some permutation groups. They give a lot of information, mostly on how these groups act on sets

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of n (i.e. $\lambda_1 + \dots + \lambda_\ell = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$).

A Ferrers diagram is a picture with λ_1 dots in the first row, λ_2 in the second row, ... and ℓ columns. The rows are written in matrix notation (from top to bottom), at least for the English notation of diagrams (French and Russian draw it differently).

Example: $\lambda = (5,4,2,1)$

Ferrers diagram of λ.



λ, with boxes instead.

Tableaux

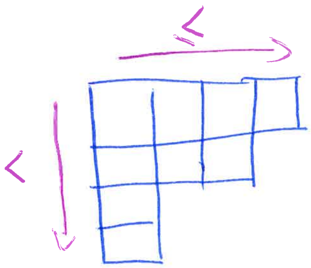
A Young tableau is a filling of the boxes of a Ferrers diagram with positive integers.

The plural of tableau is tableaux (and is pronounced in the same way).

Example

$\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ are tableaux of shape $(2,1)$ and $(2,2)$, respectively.

A <u>semistandard Young tableau</u> (SSYT) is a tableau whose entries are weakly increasing in a row, and strictly increasing on the columns.

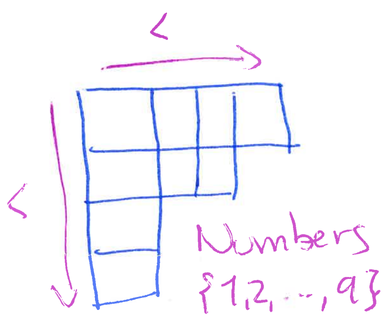


<u>Example</u>

$$\begin{array}{llll} 1 & 3 & 7 & 7 \\ 4 & 4 & 8 & \\ 5 & & & \\ 9 & & & \end{array}$$ is a SSYT.

A <u>standard Young tableau</u> (SYT) is a tableau whose entries are the numbers $\{1,2,\dots,n\}$ placed in increasing order on the row and the columns.

Equivalently, it is a SSYT with entries $[n]$.



<u>Example</u>

$$\begin{array}{llll} 1 & 3 & 6 & 7 \\ 2 & 4 & 8 & \\ 5 & & & \\ 9 & & & \end{array}$$ is a SYT (and thus a SSYT.)

$$\begin{array}{ll} 1 & 2 \\ 2 & \end{array}$$ is not a SYT.

<u>Example</u>

All the SYTs of size 2 and 3 are listed here:

$$\begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 2 \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{lll} 1 & 2 & 3 \end{array} \qquad \begin{array}{l} 1 \\ 2 \\ 3 \end{array}$$

Row-insertion (also called Schensted insertion).

Let t be a SSYT and x be a positive integer.

One can produce a new SSYT with $|t|+1$ boxes (one box more than t) and with the same content (i.e. numbers) as t, plus one occurrence of x, by row inserting x in t:

(i) If x is at least as large as all the entries on the first row of t, add x at the end of the first row.

(ii) Otherwise, find the leftmost entry of the first row that is greater than x (call it y) and replace it by x. Then, insert y in the SSYT formed by all the rows of t except the first one (by repeating this algorithm.)

The SSYT obtained in this way is denoted $t \leftarrow x$.

Example

$$t = \begin{matrix} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 & \\ 5 & 6 & & \end{matrix}, \qquad t \leftarrow 2 = \begin{matrix} 1 & 2 & 2 & \boxed{2} \\ 2 & 3 & \boxed{3} & 5 \\ 4 & 4 & \boxed{5} & \\ 5 & 6 & \boxed{6} & \end{matrix}$$

$\square$ boxes that changed: this is called the bumping route.

The row-insertion process is invertible, as long as you know which box was added:

(i) Remove the entry in that box. If it is in the first row, you are done.

(ii) Otherwise, bump the value (y) to the line above and replace the rightmost entry smaller than y by y. As long as you are not in the first row, repeat the process.

Example

$t \leftarrow 2$ from above, and the box in the fourth row.

Product SYT

The insertion operation allows us to define a product on SSYTs.

The <u>product tableau</u> $t \cdot u$ of two SSYTs is an SSYT and is obtained by successively inserting the numbers in the boxes of u in t, starting from the bottom row and, in that row, from the leftmost box.

If $u = \begin{array}{|c|c|c|} \hline u_{11} & u_{12} & u_{13} \\ \hline u_{21} & u_{22} & \\ \cline{1-2} u_{31} & & \\ \cline{1-1} \end{array}$, then $t \cdot u$ is $(((((t \leftarrow u_{31}) \leftarrow u_{21}) \leftarrow u_{22}) \leftarrow u_{11}) \leftarrow u_{12}) \leftarrow u_{13}$

Example

$$\begin{array}{llll} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 5 \\ 4 & 4 & 6 & \\ 5 & 6 & & \end{array} \cdot \begin{array}{ll} 1 & 3 \\ 2 & \end{array} = \begin{array}{llll} 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 5 \\ 4 & 4 & 5 & \\ 5 & 6 & 6 & \end{array} \cdot \begin{array}{ll} 1 & 3 \end{array}$$

$$= \begin{array}{llll} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & \\ 4 & 6 & 6 & \\ 5 & & & \end{array} \cdot \begin{array}{l} 3 \end{array}$$

$$= \begin{array}{lllll} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 5 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \\ 5 & & & & \end{array}$$

Proposition

This product makes the set of SSYTs into an associative monoid. The unit is the empty tableau.

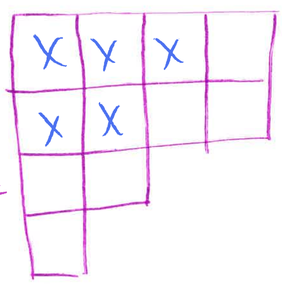
Proof: Exercise.

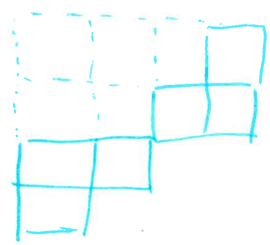
Definition

A skew diagram λ/μ is obtained from the diagrams λ and $\mu \subseteq \lambda$ (as the set of boxes) by removing from λ the boxes that also belong to μ.

Example

$(4,4,2,1)/(3,2)$

$\lambda =$  boxes in μ

$\lambda/\mu =$ 

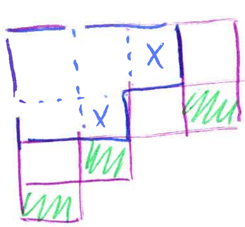
This shape is usually not left justified, hence not a diagram.

Sliding: jeu-de-taquin.

In a skew diagram, an inner corner is a box deleted from the bigger tableau such that the boxes to the right and below are not deleted. An outer corner is a box in the large diagram that has no neighbour to the right nor below.

Example

With λ and μ above, the outer corners are in green, while the inner corners are marked by 'x'.



Schützenberger's jeu-de-taquin

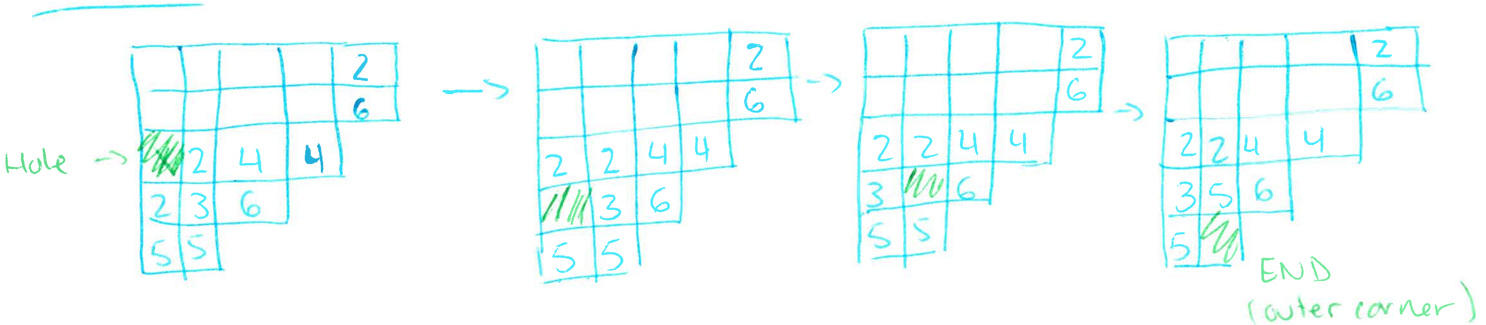
Define a skew tableau as a skew diagram filled with positive integers in a way that satisfies the rules for a SSYT i.e. weakly increasing rows, strictly increasing columns.

One can reduce the number of "holes" in the tableau by sliding non-empty boxes into empty spaces. This process is called "jeu-de-taquin" (which would be translated by the 15 puzzle).

Algorithm to fill an inner corner.

(i) The hole is the inner corner.
(ii) Slide the smallest of its two neighbors to the right and below into the hole. If the two have the same entry, slide the one below.
(iii) The hole is the box that has been slided.
(iv) Repeat (ii) and (iii) until the box is an outer corner.

Example



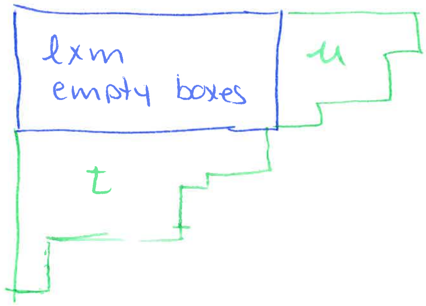
This process is reversible as long as we know the box that was removed (the hole at the end).

The rectification of a skew tableau is the successive application of the sliding until one get a SSYT. (This always appends)

Claim: Given a skew tableau, the inner corners can be filled in any order and return the same SSYT.

Product tableau

Given a tableau t with m columns and a tableau u with l rows, denote $t * u$ the skew tableau made this way:



Theorem

The rectification of $t * u$ is the product tableau $t \cdot u$, as defined on page 4.

Example: Homework VII.

Reference: William FULTON. Young tableaux. § 1.