Math 68 - Algebraic Combinatorics
Robinson-Schensted bijection

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Recall from last class that a Young tableau is standard if its entries are $\{1, 2, \dots, n\}$ and are strictly increasing from top to bottom and from left to right.

Denote f^λ the number of standard Young tableaux of shape λ.

Theorem

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Two interpretations

- In representation theory of groups, $|G| = \sum_{\substack{V_i \text{ simple} \\ G\text{-modules} \\ V_i \not\cong V_j \text{ if } i \neq j}} \dim(V_i)^2$, when G is finite and semi-simple.
  In the upcoming weeks, we will see that the simple S_n-modules are indexed by the partitions and have dimension f^λ.
- There is a bijection between pairs of SYTs with same shape and permutations.

The latter is the one that we will use for today's class

History of that theorem.

- A first bijective proof was given in 1938 by Gilbert de Beauregard Robinson.
- In 1961, Craige Schensted gave (independantly) an other proof using Schensted's insertion

Idea of the proof

- From a permutation of $[n]$, we will construct a sequence (of length n) of pairs of tableaux:

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), (P_2, Q_2), \dots, (P_n, Q_n) = (P, Q).$$

- Throughout the process, Q_i is a standard Yang tableaux. Q is thus a SYT, and is called the <u>recording tableau</u>.
- P_i is a SSYT with i distinct entries from $[n]$. Therefore, $P = P_n$ is a SYT, called the <u>insertion tableau</u>.
- We prove that the reverse algorithm is well-defined, which proves it is a bijection.

<u>Proof</u>

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}$.

The tableau $P_i = P_{i-1} \leftarrow x_i$, where $\leftarrow$ denotes Schensted's insertion

To get Q_i from Q_{i-1}, append the number i in the box $sh(P_i) / sh(P_{i-1})$.

<u>Example</u>

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{pmatrix}$

$P_1 = 4$, $P_2 = \begin{matrix} 2 \\ 4 \end{matrix}$, $P_3 = \begin{matrix} 2 & 3 \\ 4 & \end{matrix}$, $P_4 = \begin{matrix} 2 & 3 & 6 \\ 4 & & \end{matrix}$, $P_5 = \begin{matrix} 2 & 3 & 5 \\ 4 & 6 & \end{matrix}$, $P_6 = \begin{matrix} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{matrix}$, $P = P_7 = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix}$

$Q = \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix}$

Hence, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{pmatrix} \longleftrightarrow \left(\begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} , \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix} \right)$

By the way Q is constructed, Q_i and P_i always have the same shape. Also, P_i is a SSYT, because Schensted's insertion preserves the fact that rows are (weakly) increasing and columns are (strictly) increasing.

<u>Bijection</u>: From (P, Q), one can recover σ by reversing the algorithm step-by-step: $\sigma(i)$ is the value in the box that has been inserted in P at time i.
From P_i, we reverse-inverse (like on Monday) from the position of the box containing i in Q_i.

Example

$$P = \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 6 \\ 4 \end{array} \quad , \quad Q = \begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 5 \\ 6 \end{array}$$

$\boxed{}$ bumping route.

$\sigma(7)$: $\begin{array}{llll} 1 & 3 & 5 & \boxed{7} \\ 2 & 6 \\ 4 \end{array}$ (first row, hence $\sigma(7)=7$)

$\sigma(6)$: $\begin{array}{lll} \boxed{1} & 3 & 5 \\ \boxed{2} & 6 \\ \boxed{4} \end{array}$ $\sigma(6)=1$

$\sigma(5)$: $\begin{array}{lll} 2 & 3 & \boxed{5} \\ 4 & \boxed{6} \end{array}$ $\sigma(5)=5$

$\sigma(4)$: $\begin{array}{lll} 2 & 3 & \boxed{6} \\ 4 \end{array}$ $\sigma(4)=6$

$\sigma(3)$: $\begin{array}{ll} 2 & \boxed{3} \\ 4 \end{array}$ $\sigma(3)=3$

$\sigma(2)$: $\begin{array}{l} \boxed{2} \\ \boxed{4} \end{array}$ $\sigma(2)=2$

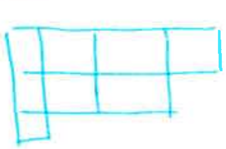
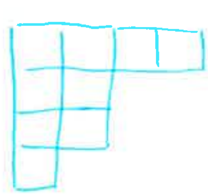
$\sigma(1)$: 4 $\sigma(1)=4$

We get

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{pmatrix}.$$

Let $\lambda \vdash n$. Its <u>conjugate</u> λ' is the partition whose Ferrers diagram is obtained by a symmetry on the main diagonal. Also, if t is a tableau of shape λ, its conjugate t' is the tableau of shape λ' obtained in the same way.

Example

 $'$ $=$  and $\begin{pmatrix} 1 & 2 \\ 3 \\ 4 \end{pmatrix}' = \begin{array}{lll} 1 & 3 & 4 \\ 2 \end{array}$

The conjugate of a SYT is a SYT.

⚠ This is not true for SSYT.

Theorem (Schützenberger, 1963)

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}$ and let $\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_1 \end{pmatrix}$

Then,

(i) $P(\sigma^{-1}) = Q(\sigma)$ and $Q(\sigma^{-1}) = P(\sigma)$

(ii) $P(\bar{\sigma}) = P(\sigma)'$.

Example

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 1 & 5 & 4 & 7 \end{pmatrix}$, $\tau = \sigma^{-1}$

$P_1 = 6$, $P_2 = \begin{array}{l} 2 \\ 6 \end{array}$, $P_3 = \begin{array}{l} 23 \\ 6 \end{array}$, $P_4 = \begin{array}{l} 13 \\ 2 \\ 6 \end{array}$, $P_5 = \begin{array}{l} 135 \\ 2 \\ 6 \end{array}$, $P_6 = \begin{array}{l} 134 \\ 25 \\ 6 \end{array}$, $P_7 = \begin{array}{l} 1347 \\ 25 \\ 6 \end{array} = P(\tau)$

So $P(\tau) = Q(\sigma)$.

Moreover, $Q(\tau) = \begin{array}{l} 1357 \\ 26 \\ 4 \end{array} = P(\sigma)$.

Example

$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 5 & 6 & 3 & 2 & 4 \end{pmatrix}$, $\tau = \bar{\sigma}$.

Then

$P_1 = 7$, $P_2 = \begin{array}{l} 1 \\ 7 \end{array}$, $P_3 = \begin{array}{l} 15 \\ 7 \end{array}$, $P_4 = \begin{array}{l} 156 \\ 7 \end{array}$, $P_5 = \begin{array}{l} 136 \\ 5 \\ 7 \end{array}$ $P_6 = \begin{array}{l} 126 \\ 3 \\ 5 \\ 7 \end{array}$, $P_7 = \begin{array}{l} 124 \\ 36 \\ 5 \\ 7 \end{array} = P(\tau)$

$P(\tau)' = \begin{array}{l} 1357 \\ 26 \\ 4 \end{array} = P(\sigma)$.

Application

Something the Robinson-Schensted bijection is useful for is to compute the number of SYTs of a given shape λ.

The application is not obvious.

Theorem (Hook formula, Frame-Robinson-Thrall, 1954)

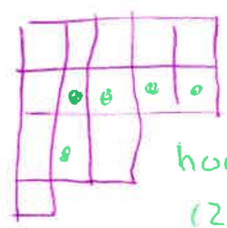
For a cell (i,j) in the diagram λ, let h_{ij} be the number of cells either

- On the same row, weakly to the right of (i,j)
- On the same column, below (i,j)

This number h_{ij} is called the hooklength of (i,j).

Then,

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}$$



hook of $(2,2)$

Example

Let $\lambda = (2,2,1) \vdash 5$.

The hook lengths are

$$\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Therefore,

$$f^{(2,2,1)} = \frac{5!}{4\cdot 3\cdot 2\cdot 1\cdot 1} = 5$$

The SYTs are

$$\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 3 & 5 \\ 4 & \end{array} \quad \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & 5 \\ 4 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}.$$

Reference: Bruce E. SAGAN. *The Symmetric Group*. §1.10, 3.1, 3.2, 3.6, 3.9, 3.10