Math 68 - Algebraic Combinatorics — Nadia Lafrenière

Increasing sequences — 10/25/2019

Let σ be a permutation of $[n]$.

The sequence $\sigma(i_1) < \sigma(i_2) < \dots < \sigma(i_\ell)$ is an increasing sequence of length ℓ if $i_1 < i_2 < \dots < i_\ell$.

Example

$\sigma = 4\,2\,3\,6\,5\,1\,7$ has increasing sequences

length	sequences
1	1, 2, 3, 4, 5, 6, 7
2	23, 26, 25, 27, 36, 35, 37, 67, 57, 46, 45, 47
3	236, 235, 237, 457, 467, 367, 357
4	2357, 2367

Try it yourself!

What is the longest increasing sequence of 3 5 6 4 8 1 7 (10) 9 2 ?

Claim: There is an easy way to know.

Theorem (Schensted, 1961)

Let $\sigma \in S_n$.

The length of the longest increasing subsequence of σ is the length of the first row of $P(\sigma)$ (its insertion tableau).

The length of the longest decreasing subsequence of σ is the length of the first column of $P(\sigma)$.

Example

From 3564817(10)92, we build the P-tableau

$$3,\quad 35,\quad 356,\quad \begin{matrix}3&4&6\\5\end{matrix},\quad \begin{matrix}3&4&6&8\\5\end{matrix},\quad \begin{matrix}1&4&6&8\\3\\5\end{matrix},\quad \begin{matrix}1&4&6&7\\3&8\\5\end{matrix},\quad \begin{matrix}1&4&6&7&(10)\\3&8\\5\end{matrix},$$

$$\begin{matrix}1&4&6&7&9\\3&8&(10)\\5\end{matrix},\quad \begin{matrix}1&2&6&7&9\\3&4&(10)\\5&8\end{matrix} = P$$

From the theorem, we know that the longest increasing sequence has length 5. Hence, 3568(10) is a longest increasing sequence.

The longest decreasing sequence has length 3, so 641 or (10)92 are suitable choices.

Proof

If the first statement is true, then the second one is true as well: since a decreasing sequence of σ is an increasing one in $\bar{\sigma}$ (its reversal), the maximal length of a decreasing sequence in σ is the length of the first row of $P(\bar{\sigma}) = (P(\sigma))'$. Hence, the length of the first column of $P(\sigma)$.

The theorem is also true if the following lemma is.

Lemma

If $\sigma = x_1 \cdots x_n$ and x_k enters → through Schensted's insertion P_{k-1} in column j, then the longest increasing sequence of σ ending in x_k has length j.

Proof of the lemma

By induction.

$k=1$: The longest increasing sequence ending in x_1 is x_1.

3

Suppose it holds up to k-1.

(i) There exists at least one increasing sequence of length j in $x_1x_2\dots x_k$. Let y be the entry at position $(1,j-1)$ in P_{k-1}. By induction hypothesis, there is a sequence ending with y of length $j-1$. Since x_k is inserted to the right of y, $x_k > y$, and we can append x_k to the sequence ending with y.

(ii) We must now prove that there is no longer increasing sequence. If there is one, there exists $i<k$ with x_i entered in a column weakly to the right of j, and with $x_i < x_k$. This way the sequence ending in x_i is still increasing when we add x_k at the end.

But since x_i is weakly to the right of x_k (let's say in column j') then $x_k \leq P(1,j') < x_i < x_k$.

$\Rightarrow\Leftarrow$

Finding the longest sequence.

We can use the last lemma to find a longest increasing sequence. Note that the items in the first row of $P(\sigma)$ are not always an increasing sequence in σ.

Example

$\sigma = 4236517$, $P(\sigma) = \begin{matrix} 1&3&5&7 \\ 2&6&& \\ 4&&& \end{matrix}$, and 1357 is not increasing in σ.

To find the longest increasing sequence:

For each i, find j_i, the column in which x_i entered P_{i-1}.

Find a number y such that x_y was entered in the last column (j_y) of P_{y-1}.

From y to 1 (downwards) look for a number z such that x_z was entered in column $j_y - 1$ of P_{z-1}.

⋮

Repeat

⋮

The sequence is the one ending with $\dots z_y$.

Example

$\sigma = 4236517$

Building $P(\sigma)$

$$4,\quad \begin{array}{l}2\\4\end{array},\quad \begin{array}{l}2\,3\\4\end{array},\quad \begin{array}{l}2\,3\,6\\4\end{array},\quad \begin{array}{l}2\,3\,5\\4\,6\end{array},\quad \begin{array}{l}1\,3\,5\\2\,6\\4\end{array},\quad \begin{array}{l}1\,3\,5\,7\\2\,6\\4\end{array}$$

i	1	2	3	4	5	6	7
x_i	4	2	3	6	5	1	7
enters in column	1	1	2	3	3	1	4

The sequence 2357 is increasing.

Question: Given σ, a permutation with longest increasing sequence of length j and longest decreasing sequence of length k, how long are the longest (increasing/decreasing) sequences in σ^{-1}?

Multiple sequences

Let π be a sequence. It is said to be <u>k-increasing</u> if it can be written (as a set) as the disjoint union of k increasing sequences.

(k-decreasing is defined the same way).

Example

$$4236517 = 2357 \sqcup 46 \sqcup 1,$$

So 4236517 is 3-increasing.

Also, 423657 is not a permutation, but it is a 2-increasing sequence.

Let $i_k(\pi)$ be the length of the longest k-increasing subsequence of π. Let $d_k(\pi)$ be the length of the longest k-decreasing subsequence of π.

Example

$$i_k(4236517) = \begin{cases} 4 & \text{if } k=1 \\ 6 & \text{if } k=2 \\ 7 & \text{if } k \geq 3. \end{cases}$$

Theorem (Greene, 1974)

Given $\sigma \in S_n$, let $sh(P(\sigma)) = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, with conjugate $(\lambda_1', \lambda_2', \dots, \lambda_m')$. Then, for any k,

$$i_k(\sigma) = \lambda_1 + \lambda_2 + \dots + \lambda_k$$
$$d_k(\sigma) = \lambda_1' + \lambda_2' + \dots + \lambda_k'.$$

Example

The P-tableau of 4236517 is
1 3 5 7
2 6
4
, so

k	i_k	d_k	examples
1	4	3	2357, 421
2	6	5	$2357 \sqcup 46$, $421 \sqcup 65$
3	7	6	$2357 \sqcup 46 \sqcup 1$, $421 \sqcup 65 \sqcup 3$
4	7	7	$421 \sqcup 65 \sqcup 3 \sqcup 7$

Reference: Bruce E. Sagan. The symmetric group §3.3, 3.5.