Math 68 - Algebraic Combinatorics
Representations and Modules

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Recall that the complex general linear group of degree d is the group of invertible matrices of dimension d over $\mathbb{C}$.

A matrix representation of a group G is a group homomorphism

$$\rho : G \to GL_d.$$

Equivalently, assing to each $g \in G$ a matrix $(d \times d)$ $\rho(g)$ such that

(i) $\rho(Id_G) = Id_d$

(ii) $\rho(gh) = \rho(g)\rho(h)$, $\forall g,h \in G$.

d is the dimension of the representation.

Examples

- All groups have the trivial representation $\rho(g) = Id$, $\forall g \in G$.
- The group C_3 also admits as representations

$$\rho_1(\bar{k}) = \begin{pmatrix} e^{\frac{2\pi i k}{3}} & 0 & 0 \\ 0 & e^{\frac{2\pi i k}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i k}{3}} \end{pmatrix}$$

and $\rho_2(\bar{k}) = \rho_1(\bar{k})^2$, with $\bar{k}$ being the class containing k, $k \in \{0,1,2\}$.

- The symmetric group admits the sign representation

$$\rho(\sigma) = sgn(\sigma)$$

- The defining representation of the symmetric group associates to σ its permutation matrix

Ex $213 \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Definition

Let V be a vector space and G be a group. Then, V is a G-module if there is a group homomorphism

$$\rho : G \to GL(V)$$

(i.e. V is stable under ρ)

Equivalently, V is a G-module if there is a multiplication gv such that

(i) $gv \in V$ (stability)

(ii) $g(cv+dw) = cg(v) + dg(w)$ (linearity) $c,d \in \mathbb{C}$, $v,w \in V$.

(iii) $g(hv) = (gh)v$ (action) $g,h \in G$.

(iv) $Id_G \cdot v = v$ (identity).

Alternatively, "V carries a representation of G".

Examples

- The vector space $\{(v_1, v_2, \dots, v_d) \mid v_1 = v_2 = \dots = v_d\}$ is a one dimensional vector space and it is fixed under S_n.

- The standard module $\{(v_1, v_2, \dots, v_d) \mid v_1 + v_2 + \dots + v_d = 0\}$ is a $(n-1)$-dimensional vector space, and it is fixed under S_n

- The vector space spanned by $\sum_{\sigma \in S_n} sgn(\sigma)\sigma$ is a one-dimensional subspace of the vector space $\mathbb{C}S_n$ (the linear combinations of permutations over $\mathbb{C}$). It is not isomorphic to the vector space spanned by $\sum_{\sigma \in S_n} \sigma$. Hence, there exists at least two one-dimensional modules on $\mathbb{C}S_n$ (with $n \geq 2$)

A submodule of V is a subspace that is invariant under the action of G.

Example

Every module V contains V and $\{\vec{0}\}$ has submodules

A nonzero module is <u>simple</u> if it does not contain a nontrivial submodule (i.e. its only submodules are $\{\vec{0}\}$ and itself)

A module is <u>indecomposable</u> if it cannot be written as a direct sum of (more than one) simple modules.

Simple ⇒ Indecomposable.

<u>Theorem</u> (Maschke, 1899)

Let G be a finite group and let V be a non-zero G-module over $\mathbb{C}$. Then, if V is indecomposable, it is simple.

Said otherwise, V can be decomposed into a direct sum of simple modules.

This theorem still holds if the base field is any field whose characteristic does not divide $|G|$.

<u>Example</u>

$\mathbb{C}S_3$ can be decomposed, since it contains $\langle \sum_{\sigma \in S_3} \sigma \rangle$ and $\langle \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\sigma \rangle$ two one dimensional modules. We will see later that

$$\mathbb{C}S_3 = \langle \sum_{\sigma \in S_3} \sigma \rangle \oplus \langle \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\sigma \rangle \oplus \underbrace{2\mathcal{U}}_{\mathcal{U} \oplus \mathcal{U}'},$$

where $\mathcal{U}$ is a 2-dimensional module. $\mathcal{U}\oplus\mathcal{U}'$, where $\mathcal{U} \cong \mathcal{U}'$.

<u>Definition</u>

Let V and W be G-modules. Then a G-<u>homomorphism</u> (or homomorphism of G-modules) is a linear transformation $\theta: V \to W$ such that $\theta(gv) = g(\theta(v))$, for all $g \in G$ and $v \in V$. (i.e. θ <u>preserves</u> or <u>respects</u> the action of G)

For such a θ, its <u>kernel</u> is $\ker \theta = \{v \in V : \theta(v) = \vec{0}\}$ and its image is $\operatorname{im} \theta = \{w \in W \mid \exists v \in V, \theta(v) = w\}$.

Proposition

Let $\theta : V \to W$ be a G-homomorphism. Then

(i) $\ker \theta$ is a G-submodule of V

(ii) $\text{im } \theta$ is a G-submodule of W.

Proof (of (i))

It means that $\ker\theta$ is stable under the action of G (we already know it's a vector space)

Let $v \in \ker \theta$. Then,

$$\begin{aligned} \theta(gv) &= g(\theta(v)) \quad \text{(since } \theta \text{ is a homomorphism)} \\ &= g \cdot \vec{0} \quad \text{since } v \in \ker\theta \\ &= \vec{0}; \end{aligned}$$

and $gv \in \ker \theta$.

Theorem (Schur's Lemma).

Let V and W be two simple G modules over $\mathbb{C}$ (or any algebraically closed base field). If $\theta : V \to W$ is a G-homomorphism, either

(i) θ is a G-isomorphism

(ii) θ is the zero map.

Proof

By the proposition, $\ker\theta$ is a G-submodule of V. Since V is simple, either $V = \ker \theta$ (and θ is the zero map) or $\ker\theta = \{\vec{0}\}$. In the latter case, $\text{im}\,\theta$ is a submodule of W, and it can't be $\{\vec{0}\}$ because θ is not the zero map. Hence, $\text{im}\,\theta = W$ and θ is surjective. But since $\ker \theta = \{\vec{0}\}$, it is also injective, and θ is an isomorphism.

Corollary (matrix version).

Let X and Y be two irreducible matrix representations (i.e. representations corresponding to simple modules) If T is any matrix such that $T\,X(g) = Y(g)\,T$ for all $g \in G$, then either

(i) T is invertible

or (ii) T is the zero matrix

Corollary

Let X be an irreducible matrix representation of G over $\mathbb{C}$. Then the only matrices that commute with $X(g)$ for all $g \in G$ are those of the form $T = cI$ (i.e. scalar multiples of the identity matrix.

Proof

By the matrix version of Schur's Lemma, $TX(g) = X(g)T$ implies that T is either invertible or zero. In the first case, write instead $TX - cX = XT - cX$, and this is $(T - cI)X = X(T - cI)$, for all $c \in \mathbb{C}$. This satisfies the matrix version, and if we take c to be an eigenvalue of T (possible because $\mathbb{C}$ is algebraically closed), $T - cI$ is not invertible, hence it is the zero map, and $T = cI$.

Corollary (rephrased)

Let V and W be two simple G-modules over $\mathbb{C}$. Then, any homomorphism $\theta : V \to W$ is a multiple of the identity.

Reference: Bruce E. Sagan. The Symmetric Group, 2nd edition. §1.2 to 1.6