Math 68 - Algebraic Combinatorics Nadia Lafrenière

Induced and restricted representations, branching rule

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Given a group G and a subgroup H of G, can we find the representations of H following these of G, and vice-versa?

Definition

Let $H \leq G$ and X be a representation of G.

Then, $X\downarrow_H^G$ is the representation defined as

$$X\downarrow_H^G(h) = X(h), \quad \forall h \in H$$

and is called the restricted representation.

How can we define the process backwards?

Is $X(g) = \begin{cases} X(h) & \text{if } h \in H \\ 0 & \text{otherwise} \end{cases}$ a representation?

Definition

Let $H \leq G$. A transversal $t_1, \dots, t_\ell$ is a set of elements of G such that any $g \in G$ can be written as $g = t_i \cdot h$ for a choice of $i \in [\ell]$ and $h \in H$.

Example

If $G = S_4$ and $H \cong S_3$ is the permutations that fix 4.

Then, $\{(), (14), (24), (34)\}$ is a transversal for S_3 in S_4.

Definition (Frobenius)

Consider $H \leq G$ and fix a transversal $t_1, \ldots, t_\ell$.

If Y is a representation of H, then the corresponding <u>induced representation</u> $Y\uparrow_H^G$ assigns to each $g \in G$ the block matrix

$$Y\uparrow_H^G(g) = \left(Y(t_i^{-1} g t_j)\right) = \begin{pmatrix} Y(t_1^{-1}gt_1) & Y(t_1^{-1}gt_2) & \cdots & Y(t_1^{-1}gt_\ell) \\ Y(t_2^{-1}gt_1) & Y(t_2^{-1}gt_2) & \cdots & Y(t_2^{-1}gt_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ Y(t_\ell^{-1}gt_1) & Y(t_\ell^{-1}gt_2) & \cdots & Y(t_\ell^{-1}gt_\ell) \end{pmatrix},$$

where $Y(g)$ is the zero matrix if $g \notin H$.

Example: $G = S_3$, $H = \{\overset{=Id}{e}, (23)\}$, $t_1 = (12)$, $t_2 = e$, $t_3 = (13)$.

Induce from the trivial representation of S_2 to S_3.

$$\mathrm{triv}\uparrow^{S_3}(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{trace: } 3$$

$$\mathrm{triv}\uparrow^{S_3}(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{trace: } 1$$

$$\mathrm{triv}\uparrow^{S_3}(13) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{trace: } 1$$

$$\mathrm{triv}\uparrow^{S_3}(23) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{trace: } 1$$

$$\mathrm{triv}\uparrow^{S_3}(123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{trace: } 0$$

$$\mathrm{triv}\uparrow^{S_3}(132) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{trace: } 0$$

Notes:
- Each block is 1×1, because triv is one-dimensional.
- The trace of the matrices is constant over a conjugacy class.
- The trace corresponds to the numbers on a given row on Wednesday's handouts.

Theorem

This construction leads to $\psi\uparrow_H^G$ being a representation of G.

Some extra theorem:

Theorem (Frobenius reciprocity - see HW 8 for definitions)

Let $H \leq G$ and suppose χ and ψ are characters of G and H, respectively. Then,

$$\langle \chi, \psi\uparrow_H^G \rangle = \langle \chi\downarrow_H^G, \psi \rangle .$$

For the symmetric group: The Branching rule.

We saw last week that we could associate bijectively partitions and non-isomorphic simple modules of the symmetric group algebra. What does that tell us about induced representations/modules?

Definition

Let λ be a diagram.

An inner corner of λ is a box in the Ferrers diagram of λ that has no neighbor on the right or below.

An outer corner of λ is a box that is not in its Ferrers diagram, but that could be added to it and it would then be an inner corner.

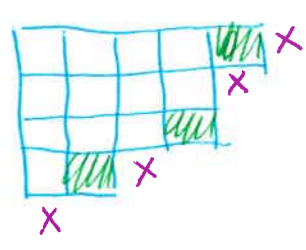
Note that this definition is not equivalent to the one I gave for skew diagrams.

Example

$\lambda = (5,4,4,2)$

outer corners: X

inner corners: ▨

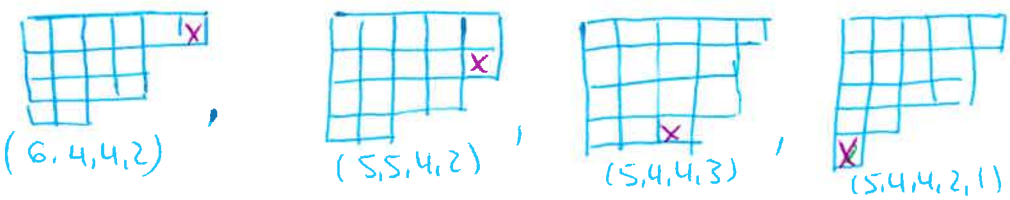


Define

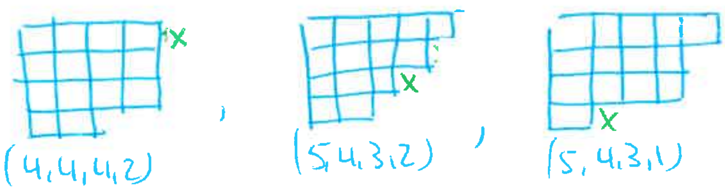
- λ^- to be the set of diagrams obtained from λ by removing an inner corner.
- λ^+, the set of diagrams obtained by adding an outer corner.

Example

$\lambda = (5,4,4,2)$

$\lambda^+ = \{$ 

$\}$.

$\lambda^- = \{$ 

$\}$.

Lemma

We have $f^\lambda = \sum_{\lambda^-} f^{\lambda^-}$.

Proof

Every SYT of shape λ is a pair of an inner corner of λ (that contains the entry n) and a SYT of shape $\lambda - \{\text{that inner corner}\}$. Conversely, all such pair is a SYT.

Theorem (Branching rule)

If $\lambda \vdash n$, then

(i) $S^\lambda \downarrow^{S_n}_{S_{n-1}} \cong \bigoplus_{\lambda^-} S^{\lambda^-}$, and

(ii) $S^\lambda \uparrow^{S_{n+1}}_{S_n} \cong \bigoplus_{\lambda^+} S^{\lambda^+}$.

Example

$$S^{(5,4,4,2)} \downarrow^{S_{15}}_{S_{14}} \cong S^{(4,4,4,2)} \oplus S^{(5,4,3,2)} \oplus S^{(5,4,4,1)}$$

$$S^{(5,4,4,2)} \uparrow^{S_{16}}_{S_{15}} \cong S^{(6,4,4,2)} \oplus S^{(5,5,4,2)} \oplus S^{(5,4,4,3)} \oplus S^{(5,4,4,2,1)}$$

Example

$$S^{triv} \uparrow^{S_3}_{S_2} \cong \underbrace{S^{\ydiagram{3}}}_{S^{triv(3)}} \oplus S^{\ydiagram{2,1}}, \qquad S^{triv(2)} = S^{\ydiagram{2}}$$

and the character of $S^{\ydiagram{2,1}}$ is the difference of those of $S^{triv} \uparrow^{S_3}_{S_2}$ and $S^{\ydiagram{3}}$

How to deduce part (ii) from part (i)? Using Frobenius reciprocity!

Fact: If χ is the character of a simple module V and ψ is a character of a module W, then $\langle \chi, \psi \rangle$ gives the number of copies of modules isomorphic to V in W.

Example: $\langle \mathbb{C}S_3, S^{\ydiagram{3}} \rangle = 1$, $\langle S^{\ydiagram{3}}, S^{\ydiagram{2,1}} \rangle = 0$ and $\langle \mathbb{C}S_3, S^{\ydiagram{2,1}} \rangle = 2$.

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We know, from part 1, that $S^\lambda \downarrow^{S_n}_{S_{n-1}} = \bigoplus_{\lambda^-} S^{\lambda^-}$, and we know that the simple modules of S_n are the Specht modules indexed by the partitions of n.

Hence, whatever $S^\mu \uparrow^{S_n}_{S_{n-1}}$ is, we can write it as

$$S^\mu \uparrow^{S_n}_{S_{n-1}} = \bigoplus_{\lambda \vdash n} m_\lambda S^\lambda .$$

multiplicity

To find the m_λ's we use Frobenius Reciprocity, along with the fact that $\langle S^\mu \uparrow^{S_n}_{S_{n-1}}, S^\lambda \rangle = m_\lambda$

↳ from last page

$$m_\lambda = \langle S^\mu \uparrow^{S_n}_{S_{n-1}}, S^\lambda \rangle$$

$$= \langle S^\mu, S^\lambda \downarrow^{S_n}_{S_{n-1}} \rangle \quad \text{By Frobenius reciprocity}$$

$$= \begin{cases} 1 & \text{if } \mu \in \lambda^- \text{ (equivalently, } \lambda \in \mu^+) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S^\mu \uparrow^{S_n}_{S_{n-1}} = \bigoplus_{\lambda \in \mu^+} S^\lambda .$$

References: Bruce E. Sagan. The Symmetric Group. § 1.12, 2.8