Math 68- Algebraic Combinatorics

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Murnaghan-Nakayama rule

The question we want to answer today is "How to compute characters for S_n"?

We saw last class that every permutation modules is obtained by induction from the trivial representation. However, there is no easy way to compute characters of induced representations, in general.

Fortunately, we also learnt how to decompose the permutation modules.

Proposition

Let X be a matrix representation of G with character χ.

Suppose $X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \cdots \oplus m_k X^{(k)}$,

where $X^{(i)}$ is a representation with character $\chi^{(i)}$.

Then,

$$\chi = m_1 \chi^{(1)} + m_2 \chi^{(2)} + \cdots + m_k \chi^{(k)}.$$

Sketch of proof

X is conjugate with the block matrix

$$\begin{bmatrix} \boxed{X_1} & & & & & & 0 \\ & \boxed{X_1} & & & & & \\ & & \boxed{\ \ \cdots} & & & & \\ & & & \boxed{X_1} & & & \\ & & & & \ddots & & \\ & & & & & \boxed{X_k} & \\ 0 & & & & & & \boxed{X_k} \end{bmatrix}$$

whose trace is $m_1 tr(X_1) + \cdots + m_k tr(X_k)$.

Example

We can now compute the character of $S^{(4,1)}$.

From last class, we know that

$$M^{(4,1)} \cong S^{(4,1)} \oplus S^{(5)},$$

and using the proposition, we get

$$\chi^{(4,1)} = \psi^{(4,1)} - \chi^{(5)}$$

χ: character of the Specht module

ψ: character of the permutation module.

$\chi^{(5)}$ is the trivial representation, so its character takes value 1 for all $\sigma \in S_5$.

cycle type	(1,1,1,1,1)	(2,1,1,1)	(2,2,1)	(3,1,1)	(3,2)	(4,1)	(5)
example	Id	(12)	(12)(34)	(123)	(123)(45)	15234	51234
#	1	10	15	20	20	30	24
$\psi^{(4,1)}$	5	3	1	2	0	1	0
$\chi^{(5)}$	1	1	1	1	1	1	1
$\chi^{(4,1)}$	4	2	0	1	-1	0	-1

To compute $\psi^{(4,1)}$, we look at how many of the tabloids of shape (4,1) are fixed by σ:

$$\begin{array}{c} \overline{1\,2\,3\,4} \\ \underline{5} \end{array} \quad \begin{array}{c} \overline{1\,2\,3\,5} \\ \underline{4} \end{array} \quad \begin{array}{c} \overline{1\,3\,4\,5} \\ \underline{2} \end{array} \quad \begin{array}{c} \overline{1\,2\,4\,5} \\ \underline{3} \end{array} \quad \begin{array}{c} \overline{2\,3\,4\,5} \\ \underline{1} \end{array}$$

Example

Character for $S^{(3,1,1)}$

We know from last lecture that

$$M^{(3,1,1)} \cong S^{(3,1,1)} \oplus S^{(3,2)} \oplus 2S^{(4,1)} \oplus S^{(5)},$$

which mans that

$$\chi^{(3,1,1)} = \psi^{(3,1,1)} - \chi^{(3,2)} - 2\chi^{(4,1)} - \chi^{(5)}$$

We know how to compute $\chi^{(5)}$ and $\chi^{(4,1)}$

We can compute $\psi^{(3,1,1)}$ by counting the number of tabloids of shape $(3,1,1)$ are fixed

1 2 3 / 4 / 5	1 2 4 / 3 / 5	1 2 5 / 3 / 4	1 3 4 / 2 / 5	1 3 5 / 2 / 4	1 4 5 / 2 / 3	2 3 4 / 1 / 5	2 4 5 / 1 / 3	2 3 5 / 1 / 4	3 4 5 / 1 / 2
1 2 3 / 5 / 4	1 2 4 / 5 / 3	1 2 5 / 4 / 3	1 3 4 / 5 / 2	1 3 5 / 4 / 2	1 4 5 / 3 / 2	2 3 4 / 5 / 1	2 4 5 / 3 / 1	2 3 5 / 4 / 1	3 4 5 / 2 / 1

Cycle type	$(1,1,1,1,1)$	$(2,1^3)$	$(2,2,1)$	$(3,1,1)$	$(3,2)$	$(4,1)$	(5)
Example	Id	(12)	$(12)(34)$	(123)	$(123)(45)$	(1234)	(12345)
#	1	10	15	20	20	30	24
$\psi^{(3,1,1)}$	20	6	0	2	0	0	0
$\psi^{(3,2)}$	10	4	2	1	1	0	0

The tabloids of shape $(3,2)$ are the following:

1 2 3 / 4 5	1 2 4 / 3 5	1 2 5 / 3 4	1 3 4 / 2 5	1 3 5 / 2 4	1 4 5 / 2 3	2 3 4 / 1 5	2 3 5 / 1 4	2 4 5 / 1 3	3 4 5 / 1 2

because $M^{(3,2)} \cong S^{(3,2)} \oplus S^{(4,1)} \oplus S^{(5)}$

Hence,
$$\chi^{(3,1,1)} = \psi^{(3,1,1)} - \left(\psi^{(3,2)} - \chi^{(4,1)} - \chi^{(5)}\right) - 2\chi^{(4,1)} - \chi^{(5)}$$
$$= \psi^{(3,1,1)} - \psi^{(3,2)} - \chi^{(4,1)}$$

So we get:

cycle type	(1,1,1,1,1)	(2,1,1,1)	(2,2,1)	(3,1,1)	(3,2)	(4,1)	(5)
example	Id	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
#	1	10	15	20	20	30	24
$\chi^{(3,1,1)}$	6	0	-2	0	0	0	1

We see it seems to be complicated to compute characters, but doable.

If we had an algorithm to do it, it should:

* Do something similar to inclusion-exclusion with characters.
* Ideally, involve pairs of partitions (one indexing the conjugacy classes, one indexing the simple representations).

Claim: That exists!

The Murnaghan-Nakayama rule

Let λ be a diagram.

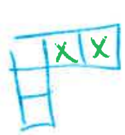
A ribbon is a skew shape λ/μ that

* does not contain $\boxplus$
* is connected (so there is a path connecting all the boxes that pass through sides of the box).

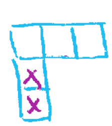
The height of a ribbon is the number of rows in λ/μ minus 1.

Example

The diagram (3,2) has two ribbons of size 2, and no ribbon of size 3 or 4.



height 0



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Theorem (Murnaghan 1937, Nakayama 1940)

The value of the character for the Specht module S^λ on permutations of cycle type $\mu = (\mu_1, \dots, \mu_\ell)$ is

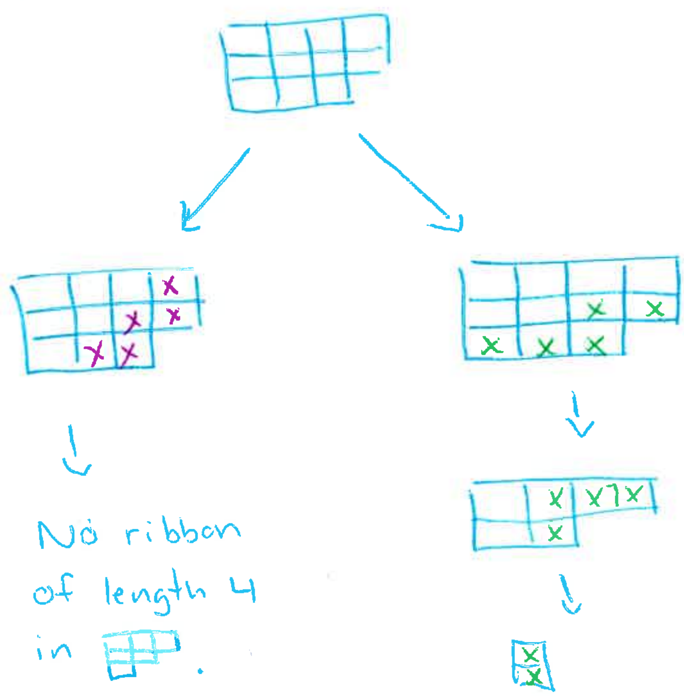
$$\chi^\lambda_\mu = \sum_{\substack{r \text{ ribbons} \\ \text{of size } \mu_1}} (-1)^{ht(r)} \, \chi^{\lambda/r}_{(\mu_2, \dots, \mu_\ell)},$$

with base case $\chi^{(0)}_{(0)} = 1$.

Example

Let $\sigma \in S_{11}$ be a permutation with cycle type $(5,4,2)$.

Then $S^{(4,4,3)}(\sigma) = -1$.



height 2

No ribbon of length 4 in $(3,2,1)$.

$\Rightarrow \chi^{(3,2,1)}(\tau) = 0$ if τ has cycle type $(4,2)$ or $(4,1,1)$

height 1

height 1

height 1

Hence, $\chi^{(4,4,3)}(\sigma) = (-1)^3 + 1 \cdot 0 = -1$.

Exercise: Check that the characters we computed for $S^{(3,1,1)}$ are right.

Corollary

All the characters of the symmetric group are integers.

Another way to find characters: tensor representation.

Theorem

Let ψ and χ be two characters.

Then,

$$\chi \otimes \psi : g \mapsto \chi(g)\psi(g)$$

is a character.

Sketch of proof

From two matrix representations φ and ρ, we get another matrix representation $\varphi \otimes \rho$ by replacing every entry of the matrix of $\varphi(g)$ by this entry multiplied by $\rho(g)$.

a scalar — a matrix.

The trace of the new matrix is $\rho(g)$ times the trace of $\varphi(g)$, hence $\psi(g)\cdot\chi(g)$.

Claim

$$S^{\lambda} \otimes S^{\begin{smallmatrix}\square\\ \vdots \\ \square\end{smallmatrix}} = S^{\lambda'}.$$

Reference: Bruce E. Sagan. *The Symmetric Group*. §1.9, 4.10.