ath 68- Algebraic Combinatorics — Nadia Lafrenière

Lecture 4 - Twelvefold way. — 09/23/2019

Counting functions between two sets

Let N and X be two sets with $\#N = n$ and $\#X = x$.
We want to count how many functions there are $N \to X$, and what happens if we add constraints.

Think of N as a set of balls, that we will place in the boxes of X. Balls could either be numbered $1, 2, \ldots, n$, or be all identical. And the boxes could also be numbered $1, 2, \ldots, x$ or all identical.

Then, there are 4 cases:

- Balls and boxes distinguishable
- Balls indistinguishable, boxes distinguishable
- Balls distinguishable, boxes indistinguishable
- Balls and boxes indistinguishable.

$|①③|_A$ $|②|_B$ and $|③|_A$ $|①②|_B$ are the same only if both balls and boxes are indistinguishable.

Moreover, we care about looking at injective, surjective or just any function.
Since there are 4 possible scenarios for the balls and the boxes, and 3 for the function, there are 12 entries to the following table:

Number of functions $N \to X$

Elements of N (balls)	Elements of X (boxes)	Any functions	Injective ones	Surjective ones
Disting.	Dist.	1 ___	2 ___	6 ___
Indist.	Dist.	11 ___	3 ___	12 ___
Dist.	Indist	8 ___	4 ___	7 ___
Indist.	Indist.	10 ___	5 ___	9 ___

- In orange: order in which we fill the table.

1 Those are the functions from $[n]$ to $[x]$.
There are x^n such functions.

2 Injective functions from $[n]$ to $[x]$.
We have x choices for the image of 1, $x-1$ choices for the image of 2, ... There are $(x)_n = x \cdot x-1 \cdot x-2 \cdot \ldots (x-n+1)$ such functions

→ decreasing factorial.

3 This is the same as 2, except that we should account for the fact that all balls are identical. Since the function is injective, there is exactly one ball in each of the n non-empty boxes, and we should divide the result of 2 by $n!$ (the permutations of the balls). This is $(x)_n/n! = \binom{x}{n}$.

4. If all boxes are identical, and there is at most one ball in each, then there is either 0 or 1 function that works, according to the pigeonhole principle. The number that goes in the table is 0 if $x < n$, and 1 otherwise.

5. This is the same as 4, since we did not care about what ball would go in what box.

6. This corresponds to the number of ways of partitioning the balls into x non-empty ordered sets.

Set compositions and set partitions

A set composition of S is an ordered grouping of the elements of S into disjoint subsets whose union is S. A composition $C_S = [B_1, \ldots, B_k]$ contains k non-empty blocks (so are the subsets called).

A set partition is an unordered partition. The partition $\pi(S) = \{B_1, \ldots, B_k\}$ contains k non-empty blocks.

Theorem

The number of set partitions of $[n]$ into k non-empty blocks is given by the following recurrence:

$$\underbrace{\#\{\pi([n]) \mid \#\pi = k\}}_{S(n,k)} = S(n-1,k-1) + k\,S(n-1,k)$$

The number $S(n,k)$ is called a Stirling number of the second kind

Proof

There are two ways of adding an element to a partition

(i) either by adding a new block
(ii) or by adding the element to an existing block.

To create a partition with k blocks, there are

- $S(n-1,k-1)$ partitions to which we can add the n^{th} element in the k-th block.
- $S(n-1,k)$ partitions with k blocks, and we can add the n-th element in either of these.

Facts about the Stirling numbers of the second kind

- $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\, i^n$
- The Bell numbers are the total numbers of partitions of n, as n is a variable:

$$B(n) = \sum_{k=0}^{n} S(n,k).$$

The entry 6 of the table is the number of partitioning $[n]$ in an ordered list of x non-empty subsets (the boxes, from 1 to x). There are $x!\,S(n,x)$ such functions

7 This is similar to 6, but we don't take into account the permutations of the boxes (they are all identical). Thus, this is the number of partitions of $[n]$ into exactly x subsets: $S(n, x)$

8. This is similar to 7, except that the function needs not to be surjective. In other words, there could be empty boxes, and we will count the partitions of $[n]$ into $1, 2, 3, \dots, x-1$ or x boxes. This is $\sum_{k=1}^{x} S(n,k)$, if n is at least one.

9. For 9 and 10, all boxes are identical, and it is the same for the balls. This means we are only interested in the decreasing list of the numbers of balls per boxes. These are the integer partitions.

For 9, the function has to be surjective, so we are looking at integer partitions of n with x parts: $P_x(n)$.

10. Since the function needs not to be surjective, there can be fewer parts in the partitions. If there is at least one ball: $\sum_{k=1}^{x} P_k(n)$.

11. Since the balls are all identical, we only want to know the number of balls in each box. This is counted by multisets (i.e. sets with repetitions), and we write the number that identifies a box every time we add a ball to it.

Example

With 4 boxes and 3 balls, the multisets are (written without parenthesis):

	111	222	333	444		$\}$ 4
112	113	114	122	133	144	$\}$ 12
233	244	223	224	334	344	
	123	234	124	134		$\}$ 4

Total: 20

Theorem

There are $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ k-multisets of $[n]$ (i.e. multisets of size k (including repetitions) with objects in $[n]$.

Example

There are $\left(\!\binom{4}{3}\!\right) = \binom{6}{3}$ ways of placing 4 balls in 3 boxes

Proof ("dots-and-bars" style)

- Take a multiset with a_i copies of i (i.e there are a_i balls in the box i)
- Rewrite this as a "word" made of bars and dots in the following way: write a_1 dots, followed by a bar, a_2 dots, ...

  This gives you a word with k dots and $n-1$ bars.

  There are $\binom{n+k-1}{k}$ such words.
- This is a bijection (you should check this is indeed true)

The number of functions for 11 is thus $\left(\!\binom{x}{n}\!\right)$

Example

Placing 3 balls into 4 boxes with 2 balls in the second and one in the third gives the word $|\cdot\cdot|\cdot|$

12 Restricting 11 to surjective functions, place one ball in each box, and then count $\left(\!\binom{x}{n-x}\!\right) = \binom{n-1}{n-x}$ ways of placing the remaining balls.

The twelvefold way

This sequence of problems is called the twelvefold way, and was created by the famous combinatorialist Gian-Carlo Rota, as a classification of twelve enumerative problems involving sets

The twelvefold way, table completed.

Number of functions $N \to X$

Elements of N (balls)	Elements of X boxes	Any function	Injective ones	Surjective ones
D	D	1 x^n	2 $(x)_n$	6 $x!\,S(n,x)$
I	D	11 $\left(\!\binom{x}{n}\!\right)$	3 $\binom{x}{n}$	12 $\left(\!\binom{x}{n-x}\!\right)$
D	I	8 $\sum_{i=0}^{x} S(n,i)$	4 $\begin{cases} 1 & \text{if } n \leq x \\ 0 & \text{if } n > x \end{cases}$	7 $S(n,x)$
I	I	10 $\sum_{i=0}^{x} p_i(n)$	5 $\begin{cases} 1 & \text{if } n \leq x \\ 0 & \text{if } n > x \end{cases}$	9 $p_x(n)$

Reference: EC1. § 1.2 (for multisets)

EC1. § 1.9 (The twelvefold way)