Math 68 - Algebraic Combinatorics   Nadia Lafrenière

Generating functions

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Definition

Let $f(n)$ be a counting function.

The ordinary generating function of f is the series (or polynomial)

$$F_{ord}(x) = \sum_{n\geq 0} f(n) x^n.$$

The exponential generating function of f is

$$F_{exp}(x) = \sum_{n\geq 0} \frac{f(n) x^n}{n!}.$$

The latter is mostly used for combinatorial problems, especially when the objects are distinguishable.

Operations on generating functions

- Addition

For f, g counting functions (of disjoint sets),

$$\sum_{n\geq 0} (f(n)+g(n)) x^n \quad \text{and} \quad \sum_{n\geq 0} \frac{(f(n)+g(n))x^n}{n!}$$

count the objects that are either in the set counted by f or in the one counted by g.

- Multiplication

$$F_{ord}(x)\,G_{ord}(x) = \sum_{n\geq 0}\left(\sum_{m=0}^{n} f(m)g(n-m)\right)x^n$$

For the ordinary generating function, the product is convolution.

Example

Use this formula to prove that

$$\frac{1}{(1-z)^k} = \sum_{n\geq 0} \underbrace{\left(\!\!\binom{k}{n}\!\!\right)}_{k \text{ multichoose } n} z^n.$$

By induction:

$k=1$.
$$\frac{1}{1-z} = \sum_{n\geq 0}\left(\!\!\binom{1}{n}\!\!\right)z^n$$
$$= \sum_{n\geq 0} z^n.$$

Assume this is true for some fixed k.

Then

$$\frac{1}{(1-z)^{k+1}} = \frac{1}{(1-z)^k}\,\frac{1}{1-z}$$

$$= \sum_{n\geq 0}\left(\sum_{m=0}^{n}\left(\!\!\binom{k}{m}\!\!\right)\cdot 1\right)x^n$$

because $[x^n]\frac{1}{1-x} = 1$.

The coefficient in front of ...

The only thing we need to prove is that $\sum_{m=0}^{n}\left(\!\!\binom{k}{m}\!\!\right) = \left(\!\!\binom{k+1}{n}\!\!\right)$

But those two things are the number of set compositions of $[n]$ in at most $(k+1)$ parts.

Multiplication (continued)

$$F_{exp}(x)\,G_{exp}(x) = \sum_{n\geq 0} \sum_{m\geq 0} \binom{n}{m} f(m)\, g(n-m) \frac{x^n}{n!}$$

Example

Recall from Homework II that $n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$, where d_{n-k} is the number of derangements of $[n-k]$

Take the exponential generating function on both sides of that equation:

$$\sum_{n\geq 0} \frac{n!}{n!} x^n = \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} d_{n-k}\, x^n$$

$$\Rightarrow \underbrace{\sum_{n\geq 0} x^n}_{\frac{1}{1-x}} = \sum_{n\geq 0} \frac{1 \cdot d_{n-k}}{n!} x^n$$

$$= \sum_{n\geq 0} \frac{x^n}{n!} \sum_{n\geq 0} \frac{d_n x^n}{n!}$$

$$= e^x \cdot D(x)$$

and the exponential generating function for the derangements is

$$D(x) = \frac{e^{-x}}{1-x}$$

Differentiation

We are allowed to differentiate the generating functions just like formal power series.

Binomial theorem

You already know that

$$(x+y)^a = \sum_{n\geq 0} \binom{a}{n} x^n y^{a-n}, \quad \text{for } a \in \mathbb{N}.$$

Evaluating at $y=1$, we get

$$(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n .$$

We can generalize this theorem to any $a \in \mathbb{Q}$:

Generalized binomial theorem

For any $a \in \mathbb{Q}$,

$$(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n,$$

where $\binom{a}{n} = \dfrac{a(a-1)(a-2)\dots(a-n+1)}{n!}$.

Solving linear recurrences

Theorem

Let $\{a_n\}_{n\geq 0}$ be a sequence. The following are equivalent:

- $\{a_n\}_{n\geq 0}$ satisfies a linear recurrence with constant coefficients $a_{n+k} + c_1 a_{n+k-1} + \dots + c_k a_n = 0$.
- Its generating function is of the form

$$\frac{P(x)}{1+c_1 x + c_2 x^2 + \dots + c_k x^k},$$

with $P(x)$ a polynomial of degree at most $k-1$.

Example

- The Fibonacci numbers have generating function

$$\frac{1}{1-x-x^2}.$$

The generalized Fibonacci numbers follow the same recurrence, but have some other initial conditions.

For example, the sequence

$$5,\ 7,\ 12,\ 19,\ 31,\ 50,\ 81,\ 131,\ \ldots$$

has generating function

$$\frac{5+2x}{1-x-x^2}.$$

The general way of solving a recurrence formula:

1. Find the set of values of the free variable ("n") for which the recurrence is true.
2. Give a name to your generating function (e.g. $F(x)$) and define it to be $F(x)=\sum_{n\geq 0} f_n x^n$
3. Multiply both sides of the recurrence by x^n, and sum over all values of n.
4. Express both sides in term of F.
5. Solve the recurrence.

Exercise: Use this technique to show that Catalan numbers are counted by

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Reference: Herbert S. WILF. generatingfunctionology, 2^{nd} edition, 1994