Math 68 - Algebraic Combinatorics

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Counting partitions

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Recall:

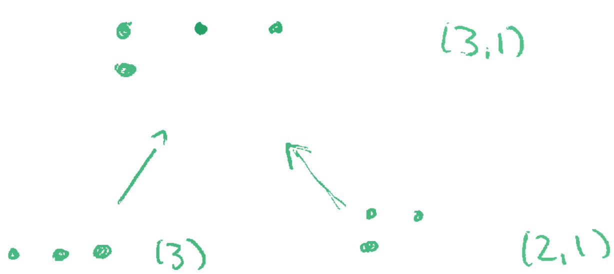
A partition of n (or integer partition), is a weakly decreasing list of positive integers whose sum is n.

Problem:

How to count the number of partitions of n?

Naive solution:

- Recursively, we can add 1 to the i-th part is the $(i+1)$-st is not the same size.
- This way, we generate all partitions.
- However, that counts multiple time the same one



- The other reason this is not very great is that we have to generate the partitions

We need more information!

Notation: Denote by $p(n)$ the number of partitions of n, and by $p_{\leq k}(n)$ the number of partitions of n with all parts of size at most k.

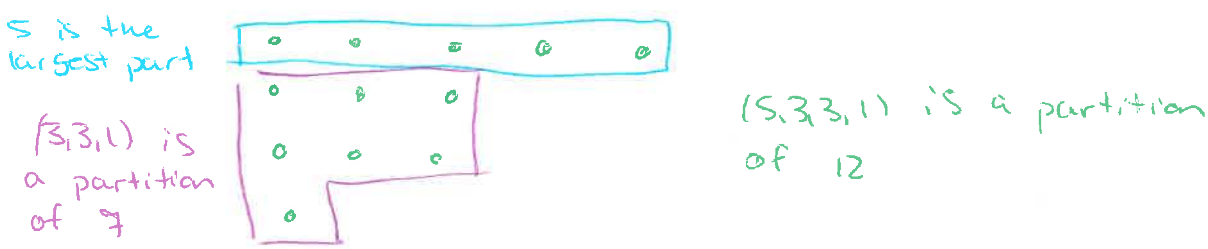
Theorem

The partitions follow this recurrence:

$$p(n) = \sum_{k=1}^{n} p_{\leq k}(n-k).$$

Proof

This sum is graded by the size of the 1^{st} part (k). What comes after is also a partition, and the total is a partition as long as all parts of the partition of $n-k$ are of length $\leq k$.



Remark

Using this recurrence, one can compute fast the number of partitions.

Counting the number of partitions of 300 on my computer took 119 ms. Of course, the implemented algorithms in Sage are much better than the one I designed and took only 1.02 ms.

Generating function

The ordinary generating function for partitions is

$$\prod_{k\geq 1} \frac{1}{1-x^k} = (1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+x^8+\ldots)(1+x^3+x^6+\ldots)\ldots$$

The coefficient of x^n when we expand the series is $p(n)$:

- Take x^{ki} from the k-th multiplicative term if there are i parts of size k.
- That should give you enough information to finish the proof.

How does that help us finding the number of partitions of n?

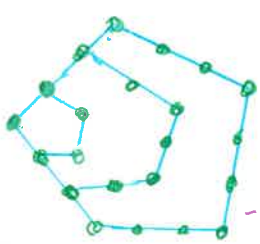
- It takes a lot of time to my computer to find the number of partitions of 300 from this method (> 1 minute).

The multiplicative inverse of this function is

$$\prod_{k\geq 1}(1-x^k) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\ldots$$

Observations on that series:

- The coefficients seem to be all -1, 0 and 1
- Apart from 0, the exponent seem to come in pairs, and the difference seem to increase by 1 every time.



- The first exponent of each pair is 1, 5, 12, 22, 51, 70, ..., which are the pentagonal numbers (see the explanation on the left.

The pentagonal numbers count the # of dots on the pentagonal net, on which there is one more dot on each edge at each level

- We can prove that the j-th pentagonal number is

$$\frac{3j^2-j}{2}.$$

Theorem (Euler)

$$\prod_{k\geq 1}(1-x^k) = 1+\sum_{j\geq 1}(-1)^j\left(x^{\frac{3j^2-j}{2}}+x^{\frac{3j^2+j}{2}}\right)$$

Using what we know about generating functions:

$$1=\prod_{k\geq 1}\frac{1-x^k}{1-x^k}=\left(\sum_{n\geq 0}p(n)x^n\right)\cdot\left(\sum_{n\geq 0}c(n)x^n\right),$$

where $c(n)$ is described by

$$c(n)=\begin{cases}-1 & \text{for } n=3j\pm j^2/2,\ j \text{ even}\\ 1 & \text{for } n=3j\pm j^2/2,\ j \text{ odd}\\ 0 & \text{otherwise}\end{cases}$$

This means that the partitions is the only sequence with

$$\sum_{k=0} c(k).p(n-k)=0,\quad \forall n\geq 1$$

Reference: Martin AIGNER and Günther ZIEGLER. Proofs from THE BOOK, 4th edition. Chapter 31