Math 68 - Algebraic Combinatorics

Incidence algebra

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Given a poset P, a (closed) interval $[s,t]$ is the induced subposet

$$\{u \mid u \geq s \text{ and } u \leq t\}.$$

A poset is said to be locally finite if all its intervals are finite.

E.g. $(\mathbb{N}, \leq)$ is an infinite, but locally finite poset

All finite posets are locally finite

Definition

Let P be a locally finite poset, and $\mathrm{Int}(P)$ the set of its intervals.

The incidence algebra of P $I(P)$ is the set of all functions $f: \mathrm{Int}(P) \to \mathbb{R}$ under the operations of

- addition

$$(f+g)\underbrace{(x,z)}_{\hookrightarrow \text{ interval } [x,z]} = f(x,z) + g(x,z)$$

- scalar multiplication, with $c \in \mathbb{R}$

$$(c \cdot f)(x,z) = c\,(f(x,z)).$$

- convolution product

$$(f * g)(x,z) = \sum_{x \leq y \leq z} f(x,y)\, g(y,z).$$

Theorem

If P is a locally finite poset, $I(P)$ is an associative algebra.

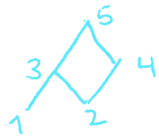
The proof is straightforward and not very enlightning. However we should point out what the identity element is for the convolution:

$$\delta(x,z) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise.} \end{cases}$$

Remark:

The incidence algebra is isomorphic to the algebra of real upper triangular matrices that have $m_{xz}=0$ whenever $[x,z]$ is not an interval.

E.g.



$$\begin{bmatrix} * & 0 & * & 0 & * \\ 0 & * & * & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

If you choose all the non-zero values of the matrix to be 1's, you get the incidence matrix of the directed graph obtained by drawing an edge from a to b if $a \leq b$.

Proposition

Let $f \in I(P)$. The following conditions are equivalent:

(i) f has a left inverse

(ii) f has a right inverse

(iii) f has a two-sided inverse

(iv) $f(t,t) \neq 0 \quad \forall t \in P$.

Moreover, if f^{-1} exists, then $f^{-1}(s,t)$ depends only on $[s,t]$.

Proof

(iii) $\Rightarrow$ (iv) If $fg=\delta$, then $f(s,s)\,g(s,s)=1$, $\forall s \in P$.

(iv) $\Rightarrow$ (ii) Since $f(t,t) \neq 0 \; \forall t \in P$, we can define the right-inverse of f. You can check that

$$g(s,u) = -f(s,s)^{-1} \sum_{s<t\leq u} f(s,t)\,g(t,u)$$

is the right inverse of f, and only depends on $[s,u]$.

(iv) $\Rightarrow$ (i) We can in a similar way define a left-inverse, h.

(i)+(ii) $\Rightarrow$ (iii) If $hf=\delta=fg$, then $f=g^{-1}$ and $f=h^{-1}$, and so $g=h$ is the unique two-sided inverse.

Some elements of the incidence algebra

- The zeta function ζ, defined by

$$\zeta(t,u) = 1 \quad \text{for all } t \leq u \text{ in } P.$$

- ζ^2:

$$\zeta^2(s,u) = \sum_{s \leq t \leq u} 1 = \#[s,u].$$

- or more generally:

$$\zeta^k(s,u) = \sum_{s = s_0 \leq s_1 \leq \dots \leq s_{k-1} \leq s_k = u} 1$$

is the number of multichains (i.e. chains with repeated elements) from s to u.

- Similarly,

$$(\zeta - 1)(s,u) = \begin{cases} 1 & \text{if } u > s \\ 0 & \text{otherwise.} \end{cases}$$

- and

$$(\zeta - 1)^k (s,u)$$

is the number of (regular) chains of length k from s to u.

- Consider $2 - \zeta = 2\delta - \zeta$:

$$(2 - \zeta) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s < t \end{cases}$$

By the proposition from last page, $2 - \zeta$ is invertible.

<u>Claim</u>: $(2-\zeta)^{-1}$ is the total number of chains.

<u>Sketch of proof</u>: Let ℓ be the length of the longest chain in the interval $[s,u]$. Then, $(\zeta - 1)^{\ell+1}(t,v) = 0$ for all $s \leq t \leq v \leq u$. Thus, we have

$$(2-\zeta)[1 + (\zeta-1) + (\zeta-1)^2 + \dots + (\zeta-1)^\ell](t,v)$$
$$= [1 - (\zeta - 1)][1 + (\zeta-1) + (\zeta-1)^2 + \dots + (\zeta-1)^\ell](t,v)$$
$$= [1 - (\zeta-1)^{\ell+1}](t,v) = \delta(t,v).$$

Hence, $(2-\zeta)^{-1} = 1 + (\zeta - 1) + \dots + (\zeta-1)^\ell$ is the total number of chains in the interval.

- eta function.

$$\eta(s,t) = \begin{cases} 1 & \text{if } t \text{ covers } s \\ 0 & \text{otherwise.} \end{cases}$$

- and

$$(1-\eta)^{-1}(s,t)$$

is the number of maximal chains in the interval $[s,t]$.

Proof : exercise.

- Möbius function

The inverse of the zeta function is the Möbius function, defined as

$$\mu(s,t) = \begin{cases} 1 & \text{if } s \leq t \\ -\sum\limits_{s \leq u < t} \mu(s,u), & \text{otherwise.} \end{cases}$$

To be able to prove it, consider the following lemma.

Lemma

$$\sum_{x \leq y \leq z} \mu(x,y) = \delta_{x,z}.$$

Proof

By definition of μ, if $x=z$, then $\sum \mu(x,y) = \mu(x,x) = 1$.

Otherwise,
$$\sum_{x \leq y \leq z} \mu(x,y) = \sum_{x \leq y < z} \mu(x,y) + \mu(x,z)$$
$$= \sum_{x \leq y < z} \mu(x,y) - \sum_{x \leq y < z} \mu(x,y)$$
$$= 0.$$

Proof of the inverse

$$(\mu * \zeta)(x,z) = \sum_{x \leq y \leq z} \mu(x,y)\,\zeta(y,z) \overset{\text{by definition of } \zeta}{=} \sum_{x \leq y \leq z} \mu(x,y) = \delta_{x,z}$$

Chains in distributive lattices

Proposition (3.5.1 and 3.5.2 in [EC1])

Let P be a finite poset and $m \in \mathbb{N}$.

The following quantities are equal

a. The number of order-preserving maps $\sigma : P \to [m]$ (resp. surjective order-preserving ...)

b. The number of multichains (resp. chains) of length m in $J(P)$.

Since all finite distributive lattices are isomorphic to some order ideals poset $J(P)$, we have more interpretations of what ζ^k and $(\zeta-1)^k$ mean.

Why is that important?

Mostly because one special type of map in a), that are the linear extensions, or bijective order-preserving maps.

Example There are 2 bij. order-preserving maps of

$\emptyset, \{1\}, \{2\}, \{1,2\}$ and $\emptyset, \{2\}, \{1\}, \{1,2\}$

References: [AOC] Bruce E. SAGAN. Combinatorics, the art of Counting. §5.4 and 5.5.

[EC1] Richard P. STANLEY. Enumerative Combinatorics, Volume 1. §3.5, 3.6 and 3.7