Mathematics 69
Winter 2011
Final Paper Assignment

Due Dates

Monday, February 21 — Final paper assigned.
Friday, February 25 — Preliminary questions due in class.
Monday, March 7 — First draft due in class. Draft will be graded Credit/No Credit based on completeness.
Wednesday, March 9 — First drafts returned.
Monday, March 14 — Final Draft due at my office (Kemeny 330) or electronically by 3pm.

Synopsis

Your goal for this assignment will be twofold. First, you will connect what you have learned in this class with a topic outside of logic. This may include using first order logic to prove something in a different field, using some other branch of mathematics to prove a fact about logic, or both. Second, you will write a formal, properly referenced, mathematical paper.

There are five different topics included in this packet. You will choose one to write your paper on. Each topic includes a series of questions that you will need to answer in your writeup. You should read each of the problems carefully and choose the one that interests you the most. Next, write up solutions to the "preliminary" questions and turn them in by Friday, February 25. These questions are meant to give you an indication of what will be involved in completing the assignment proper. If the preliminary questions seem terribly difficult you might consider choosing a different topic. Next, you should write up proofs for the main portion of the assignment. These proofs are going to form the bulk of your paper. I will be available during office hours or by appointment if you need help solving your chosen problem. You should definitely start working on your paper before your preliminary questions are graded and returned.

The style of this paper should be similar to the style of an expository article in a mathematics journal, which is not the same as either an essay for an English class or a homework paper for a math class. If you look on the course web site, on the "General Information" page, under "Exams," you will find a number of resources that you may find helpful. To get an idea of what a mathematics paper looks like you could also go to the Cook Math Collection on the third floor of Berry Library and peruse some of the articles in Mathematics Magazine. The basic idea is that your paper will include all the usual components of an essay, incorporated with mathematical proofs written in an expository format using full sentences. Another important facet is that the paper must be properly referenced, something which is discussed below. Your first draft will be due next Monday, March 7. I will proofread your writeup and have it back to you by Wednesday, March 9. This draft will be graded credit/no credit by completeness. That is, I will comment on, but not grade based on, the correctness of your proofs or your grammar, and I will give you full credit as long as you have included references and all the questions in the assignment are answered in some form. In other words it needs to be a "true" first draft. The final draft will be graded based on the quality of your proofs and your writing and is due by 3pm on Monday, March 14. It should be turned in electronically or at my office in Kemeny 330.

Honor Code and Citing Sources

There are some parts of this assignment that you may discuss with other students or with me; the problems detail which portions those are. However, if you do so, you must acknowledge any help you get. Your acknowledgment should appear in your paper as a citation and on your References page as a source. It should not appear as an attached note addressed to me. If you are not sure about how to cite sources or list them on your References page, you can consult the booklet Sources, which is available online, or me.

Do not forget that whenever you use another person's words you must indicate that you are using a direct quotation. This applies to formulas as well. Generally we do not put quotation marks around formulas, whatever their source. Often we display formulas as we would a lengthy quotation, but we do that with our own formulas as well as others'. Therefore, it is important to state clearly that you have taken a specific formula from a particular book, article or other source (See the Compactness Theorem example below.) Well-known facts that were known to you before you started writing this paper (this includes pretty much everything we've learned about first-order logic) generally do not have to be cited, but specific ways of phrasing them do, and whenever possible you should give credit to the people who proved theorems. For example, the Completeness Theorem and the Compactness Theorem are well-known. Your paper could say, without further attribution:

> Theorem 1 (Gödel's Completeness Theorem). Every consistent set of formulas is satisfiable.

However, if you need to look up the statement of the Compactness Theorem in Enderton, or if you use Enderton's way of phrasing the Compactness Theorem, you should give appropriate credit:

> Theorem 2. Two equivalent versions of the Compactness Theorem, as stated in Enderton's textbook [1, p. 144], are:
>
> 1. If $\Gamma \models \varphi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \models \varphi$.
> 2. If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Other main theorems in Enderton can similarly be treated as common knowledge, but the names Enderton uses for theorems are sometimes specific to the textbook. The Completeness Theorem, the Compactness Theorem, and the theorems named after individuals generally go by those names. However, if you have occasion to refer to the "Enumeration Theorem," for example, you should not only cite Enderton but also tell the reader what the Enumeration

Theorem says. Similarly, Enderton's notation is not necessarily standard. if you are making significant use of that notation it would be appropriate to, at the very least, give a reference for your notation so your reader knows where to go to figure out what you're saying. (I say "at the very least" because you could also explain the notation in your paper. You can also use English words in place of formal notation, and you should do so whenever the English is clearer.)

All these considerations apply to any results you use from outside of class. For example, the fact that any two vector spaces with bases of the same cardinality are isomorphic can be used without reference. However, if you actually look up and use a theorem in one of your textbooks you should reference it. If you are in doubt, please ask. And remember, it is better to include an unnecessary citation than to leave out a required one. Finally, first drafts of papers, even though they are drafts, are being submitted as your work and should include all appropriate citations and a References page.

Formatting, Etc.

Your paper should be typed on one side of the paper, double-spaced, with reasonable margins, and with your name and the page number on every page. Don't use too small a font, please (12 point is okay). While it is acceptable to write in formulas and special symbols by hand, using an equation editor is preferable. In any case, make sure any hand-written additions are legible. If you haven't been using LaTeXfor your homework, this might be a good time to learn. Pronounced la-tek, LaTeXis a typesetting program used to write up the vast majority of modern mathematics. Once again, I will be available during office hours and by appointment.

Problem 1. Vector Spaces over the Rationals

The goal for this problem is to describe a language and a set of axioms for vector spaces over the rationals, to explore what kind of statements we can make about rational vector spaces using first order logic, and to investigate the completeness of our theory. The reason we are using vector spaces over $\mathbb{Q}$ is that this will allow us to use a countable language. The prerequisite for this problem is an understanding of basic axiomatic vector space theory. Before you begin to work on the assignment proper you should write up answers to the following preliminary questions.

Preliminary questions

1. Define a language $\mathcal{L}$ for vector spaces over the rational numbers. You should implement scalar multiplication by including a one place function symbol for each rational number.

2. Describe a set Σ of sentences of $\mathcal{L}$ such that any structure that satisfies Σ is a vector space over the rationals. Hint: Σ will be infinite; be aware that you cannot quantify over the rational numbers.

3. Let $\mathfrak{A}$ be the vector space $\mathbb{Q}^2$ with the usual vector addition and scalar multiplication. Recall that automorphisms of $\mathfrak{A}$ (in our sense) are just vector space automorphisms and that any such function can be produced by mapping one basis of $\mathfrak{A}$ onto another (or the same) basis of $\mathfrak{A}$. What elements of $\mathbb{Q}^2$ are definable in $\mathcal{L}$? What subsets of $\mathbb{Q}^2$ are definable? Hint: Suppose v and w are two different vectors in $\mathfrak{A}$; when is there an automorphism taking v to w?

Assignment

Your paper should do (at least) the following six things. It is all right to discuss the first four items with other students or with me if you have trouble with them, remembering to acknowledge any help you get. Items 5 and beyond you should not discuss with anyone else.

1. Describe a language $\mathcal{L}$ for vector spaces over $\mathbb{Q}$ and give a set of sentences Σ such that a structure $\mathfrak{A}$ for $\mathcal{L}$ is a vector space over $\mathbb{Q}$ if and only if $\mathfrak{A}$ is a model of Σ.

2. Show that for every n there is a set of formulas Γ_n such that a model $\mathfrak{A}$ of Σ has dimension greater than or equal to n if and only if there is a variable assignment s for $\mathfrak{A}$ such that $\mathfrak{A}$ with s satisfies Γ_n. Show that there is a set of formulas Γ_∞ such that a model $\mathfrak{A}$ of Σ has infinite dimension if and only if there is a variable assignment s for $\mathfrak{A}$ such that $\mathfrak{A}$ with s satisfies Γ_∞.

3. Show that $Cn\,\Sigma$ is not a complete theory; give a sentence σ and show that $\Sigma \not\models \sigma$ and $\Sigma \not\models \neg\sigma$. (Hint: The zero-dimensional vector space has special properties.)

4. Show that $\Sigma \cup \{\sigma\}$ and $\Sigma \cup \{\neg\sigma\}$ both give us complete theories. Hint: Show that if $\mathfrak{A}$ is a model for Σ with non-zero dimension then $Th\,\mathfrak{A} \cup \Gamma_\infty$ is satisfiable. Then show that any two countable vector spaces over $\mathbb{Q}$ with infinite dimension are isomorphic. Alternatively, you could show that the theory of non-zero vector spaces over $\mathbb{Q}$ is categorical in the cardinality of the reals.

5. What would have happened if you picked a different sentence τ such that $\Sigma \not\models \tau$ and $\Sigma \not\models \neg\tau$ in part 3? Would you have gotten different theories in part 4? How many theories extending $Cn\,\Sigma$ are there?

6. Discuss the consequences of these results. Is there an effective procedure for determining whether a sentence of $\mathcal{L}$ is a consequence of Σ? Are there sentences of $\mathcal{L}$ whose truth or falsity in a vector space can be used to determine something about the dimension of that vector space? Are there other important conclusions you can come to?

7. Optional: what happens if, instead of rational vector spaces, we consider vector spaces over a finite field $\mathbb{F}$? You can consider finite fields in general or pick a specific one.

Problem 2. Abelian Groups

The goal for this problem is to describe a language and a set of axioms for abelian groups, and to explore what the theory of abelian groups can tell us about completeness. Prerequisite for this problem is an understanding of axiomatic group theory and axiomatic vector space theory. Before you begin work on the assignment proper you should write up answers to the following preliminary questions.

For this problem we will use the following definitions. An element g of a group G has *order n* if

$$ng = \underbrace{g + \cdots + g}_{n \text{ times}} = 0.$$

For example, g has order 2 if g + g = 0. An element is said to be *torsion free* if it does not have order n for any $n \in \mathbb{N}$ with $n > 0$. A group is said to be *torsion free* if each of its elements, other than the identity, is torsion free. Lastly, we say that a group G is divisible if for each $g \in G$ and $n \in \mathbb{N}$ with $n > 0$ there exists $h \in G$ such that

$$nh = \underbrace{h + \cdots + h}_{n \text{ times}} = g.$$

Preliminary Questions

1. Define a language $\mathcal{L}$ and a set of axioms Σ such that any model that satisfies Σ is an abelian group. Next, define a set of axioms T so that any model which satisfies $\Sigma \cup T$ is a divisible torsion free abelian group.

2. Show that any divisible torsion free abelian group has a $\mathbb{Q}$-vector space structure. Hint: Show that if G is such a group, $n \in \mathbb{N}$ with $n > 0$ and $g \in G$ then there is a unique $h \in G$ such that $nh = g$.

3. Define a set of set of axioms S such that any model that satisfies $\Sigma \cup S$ is an abelian group in which each element other than the identity has order two. Can we give a model for $\Sigma \cup S$ a vector space structure? Hint: Be creative in your choice of the scalar field.

Assignment

Your paper should do (at least) the following four things. It is all right to discuss the first three items with other students or with me if you have trouble with them, remembering to acknowledge any help you get. Items 4 and beyond you should not discuss with anyone else.

1. Define a language $\mathcal{L}$ for groups and give a set of sentences Σ such that a structure $\mathfrak{A}$ for $\mathcal{L}$ is an abelian group if and only if it is a model of Σ.

2. Define a set of sentences T such that a structure $\mathfrak{A}$ for $\mathcal{L}$ is a model for $\Sigma \cup T$ if and only if it is a divisible torsion free abelian group. Show that $\Sigma \cup T$ is categorical in the cardinality of the reals but not countably categorical. Hint: You will need to view a divisible torsion free abelian group as a vector space over the rationals.

3. Define a set of sentences S such that a structure $\mathfrak{A}$ for $\mathcal{L}$ is a model for $\Sigma \cup S$ if and only if $\mathfrak{A}$ is an abelian group in which every element other than the identity has order 2. Show that $\mathfrak{A}$ is countably categorical.

4. Discuss the consequences of these results. Is $Cn\,\Sigma$ complete? Is $Cn\,\Sigma \cup T$ complete? Is it decidable? Is $Cn\,\Sigma \cup S$ complete? If not, then can you find a set of sentences R such that $\Sigma \cup S \cup R$ has a complete theory with the same infinite models as $\Sigma \cup S$? What other important conclusions can you come to?

5. Optional: Show that any torsion free abelian group can be ordered so that

$$a < b \wedge c \leq d \Longrightarrow a + c < b + d.$$

Problem 3. Random Graphs

The goal for this problem is to use the mechanics of first order logic to explore the theory of random finite graphs and prove that the "almost sure" theory of graphs is complete and decidable. A knowledge of graphs is not necessary to solve this problem but may help to motivate some of the questions. A basic knowledge of probability, however, is a prerequisite. Before you begin work on the assignment you should write up the answers to the following preliminary questions.

For this problem we will define a *graph* to be a set G of vertices along with a relation R on G which tells us if there is an edge between two vertices. We require that the edges be "undirected" and that no vertex has an edge back onto itself. In the interests of being a specific as possible, let $\mathcal{L}$ contain a single binary relation symbol R. Our graph axioms are then given by $\forall x \neg Rxx$ and $\forall x \forall y (Rxy \to Ryx)$. Any structure for $\mathcal{L}$ satisfying these axioms is a graph. Next, for each $n \in \mathbb{N}$ with $n > 0$, let φ_n be the "extension axiom"

$$\forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} x_i \neq y_j \to \exists z \bigwedge_{i=1}^{n} (x \neq x_i \wedge z \neq y_i \wedge Rx_i z \wedge \neg Ry_i z) \right).$$

Lastly, we define Σ to be the set

$$\{\forall x \neg Rxx, \forall x \forall y (Rxy \to Ryx), \exists x \exists y \, x \neq y\} \cup \{\varphi_n \mid n = 1, 2, 3, \ldots\}.$$

Preliminary Questions

1. Show that a model of Σ is a graph where for any finite disjoint sets of vertices X and Y we can find a vertex not contained in X or Y with edges going to every vertex in X and no vertex in Y .

2. Show that there is a countable model of Σ. Hint: First show that given any countable graph G there is a graph $\overline{G}$ such that $\overline{G}$ is countable, $\overline{G}$ contains G as a subgraph, and if X and Y are disjoint finite subsets of G then there is a $z \in \overline{G} - G$ such that for all $x \in X$, Rxz, and for all $y \in Y$, $\neg Ryz$.

3. Suppose we construct a graph with vertices $\{1, 2, ..., N\}$ by independently deciding whether there is an edge between i and j for $i \neq j$ with probability $\frac{1}{2}$. Let $\mathcal{G}_N$ be the set of all graphs with vertices $\{1, 2, ..., N\}$. What is the probability that we have constructed any particular element of $\mathcal{G}_N$?

Assignment

Your paper should do (at least) the following five things. It is all right to discuss the first three items with other students or with me if you have trouble with them, remembering to acknowledge any help you get. Items 4 and beyond you should not discuss with anyone else.

1. Show that a model for Σ is a graph where for any finite disjoint sets X and Y we can find a vertex not contained in X or Y with edges going to every vertex in X and no vertex in Y. Show that Σ is satisfiable and countably categorical. Is $Cn\,\Sigma$ complete and decidable? Hint: You already showed that Σ is satisfiable. Use an argument similar to the proof that countable dense linear orderings without endpoints are countably categorical to show that Σ is countably categorical. In particular, given two models, list the elements of each and build your isomorphism one element at a time.

2. Let $\mathcal{G}_N$ be the set of all graphs with vertices $\{1, 2, ...N\}$. Consider a probability distribution on $\mathcal{G}_N$ that makes all graphs equally likely. For any sentence ψ let $p_N(\psi)$ denote
$$\frac{|\{G \in \mathcal{G}_N \ : \ G \models \varphi\}|}{|\mathcal{G}_N|},$$
the probability that a random element of $\mathcal{G}_N$ satisfies ψ. Show that
$$\lim_{N\to\infty} p_N(\varphi_N) = 1 \text{ for } n = 1, 2, 3, \ldots$$
Hint: This argument mostly relies on probability theory. You may find it easier to show that
$$\lim_{N\to\infty} p_N(\neg\varphi_N) = 0.$$

3. For any sentence ψ, show that either $\lim\limits_{N\to\infty} p_N(\psi) = 1$ or $\lim\limits_{N\to\infty} p_N(\psi) = 0$.

4. We call $T = \{\varphi \ : \ \lim\limits_{N\to\infty} p_N(\psi) = 1\}$ the almost sure theory of graphs. Show that Σ axiomatizes the almost sure theory of graphs.

5. Discuss the consequences of these results. Is T complete? Decidable? How do you interpret the fact that, for each ψ, $\lim\limits_{N\to\infty} p_N(\psi)$ is either zero or one, as a statement about graph theory? What other important conclusions can you come to?

Problem 4. Dense Linear Orderings

This goal for this problem is to use the theory of dense linear orderings to prove interesting facts about first order logic. In order to solve this problem you need to be comfortable with the reals, the rationals, and the concept of a least upper bound. Before you begin work on the assignment you should write up the answers to the following preliminary questions.

Preliminary Questions

1. Write down a language $\mathcal{L}$ and a set of axioms Σ for dense linear orderings without endpoints. We showed in class that Σ is countably categorical. Write down an outline of that proof.

2. A set D in a model $\mathfrak{A}$ for Σ is *dense* if for all $x < y$ in $|\mathfrak{A}|$ there exists $z \in D$ such that $x < z < y$. A set is *codense* if its complement is dense. Give an example of a dense, codense set in $\mathbb{Q}$. Suppose we add the unary predicate P to $\mathcal{L}$. Write down a sentence that will guarantee that P is given by a dense, codense set.

3. Prove the following two general results about first order logic.

   (a) For any language $\mathcal{L}$, two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if they are elementarily equivalent for every finite sublanguage.

   (b) If $\mathcal{L}$ is countable, T is an $\mathcal{L}$-theory with no finite models, and any two countable models of T are elementarily equivalent, then T is complete.

Assignment

Your paper should do (at least) the following four things. It is all right to discuss the first three items with other students or with me if you have trouble with them, remembering to acknowledge any help you get. Item 4 and beyond you should not discuss with anyone else.

1. Write down a language $\mathcal{L}$ and give a set of axioms Σ such that a structure for $\mathcal{L}$ is a model for Σ if and only if it is a dense linear ordering without endpoints.

2. Let $\mathcal{L}_3$ be $\mathcal{L}$ with the added constants symbols $c_0, c_1, \ldots$. Let Σ_3 be Σ with sentences asserting $c_0 < c_1 < \cdots$. Show that Σ_3 has exactly three countable models up to isomorphism. Hint: Consider the questions: Does $c_0, c_1, c_2, \ldots$ have an upper bound? a least upper bound?

3. Let $\mathcal{L}_4$ be $\mathcal{L}_3$ with the unary predicate P. Define a sentence σ that guarantees that for any model $\mathfrak{A}$ of Σ we have $\mathfrak{A} \models \sigma$ if and only if $P^{\mathfrak{A}}$ is a dense codense subset of $\mathfrak{A}$. Let
$$\Sigma_4 = \Sigma_3 \cup \{\sigma\} \cup \{\neg P c_i\}_{i=0}^{\infty}.$$
Show that Σ_4 has exactly four countable models up to isomorphism.

4. Discuss the consequences of these results. Are $Cn\,\Sigma_3$ and $Cn\,\Sigma_4$ complete? Are they decidable? Can you generalize this construction to give examples of complete theories which have exactly n countable models for $n = 5, 6, \ldots$? What other important conclusions can you come to?

5. Optional: show that Σ is not categorical in the cardinality of the reals.

Problem 5. Game Theory

The goal for this problem is to use techniques from game theory to prove facts about models in first order logic. We will use Ehrenfeucht-Fraïssé games to prove that the theory of discrete linear orderings without end points is complete. A prerequisite for this problem is a rather basic knowledge of game theory. Before you begin work on the assignment you should write up the answers to the following preliminary questions.

We use the following two definitions in this problem. First, suppose we have a game G with two players named Alice and Bob respectively. A *strategy* for Bob is a function τ such that if Alice's first n moves are $c_1, \ldots, c_n$ then Bob's n^{th} move will be $\tau(c_1, \ldots, c_n)$. We say that Bob uses the strategy τ if the play of the game looks like:

$$\text{Alice: } c_1; \text{ Bob: } \tau(c_1); \text{ Alice: } c_2; \text{ Bob: } \tau(c_1, c_2); \ldots$$

We say that τ is a *winning strategy* for Bob if for any sequence of plays $c_1, c_2, \ldots$ that Alice makes, Bob will win by following τ. We define winning strategies for Alice analogously. Next, let $\mathcal{L}$ be a language with no function symbols. Suppose we have two structures $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathcal{L}$ with $|\mathfrak{A}| \cap |\mathfrak{B}| = \emptyset$. If $A \subseteq |\mathfrak{A}|$ and $B \subseteq |\mathfrak{B}|$ and $f : A \to B$ we say that f is a *partial embedding* if the function

$$f \cup \{(c^{\mathfrak{A}}, c^{\mathfrak{B}}) \ : \ c \text{ is a constant in } \mathcal{L}\}$$

is a bijection preserving all relations of $\mathcal{L}$. We will define an infinite two-player game $G_\omega(\mathfrak{A}, \mathfrak{B})$ between two players called Alice and Bob. A play of the game will consist of a (countably) infinite number of stages. Together they will build a partial embedding f from $\mathfrak{A}$ to $\mathfrak{B}$. At the i^{th} stage, Alice moves first and either plays $a_i \in |\mathfrak{A}|$, challenging Bob to put a_i into the domain of f, or $b_i \in |\mathfrak{B}|$, challenging Bob to put b_i into the range of f . If Alice plays a_i then Bob must play $b_i \in |\mathfrak{B}|$, whereas if Alice plays b_i then Bob must play $a_i \in |\mathfrak{A}|$. Bob wins the game if $f = \{(a_i, b_i) \ : \ i = 1, 2, \ldots\}$ is the graph of a partial embedding. Finally, we say that a linear order is discrete if each element has an immediate successor and predecessor.

Preliminary Questions

1. Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable $\mathcal{L}$ structures then Bob has a winning strategy in $G_\omega(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

2. Reconstruct our proof that any two countable dense linear orderings without endpoints are isomorphic in terms of the game $G_\omega(\mathfrak{A}, \mathfrak{B})$.

3. Define a language $\mathcal{L}$ and a set of axioms Σ for the theory of discrete linear orders without endpoints. Show that every model of Σ is of the form $(L \times \mathbb{Z}, <)$ where L is a linear order and $<$ is the lexicographic order ($(x, m) < (y, m)$ if either $x < y$, or $x = y$ and $m < n$). Also show that every order of this form is a model of Σ. Hint: consider the equivalence relation $a \equiv b$ if and only if there are finitely many elements between a and b.

Assignment

Your paper should do (at least) the following five things. It is all right to discuss the first four items with other students or with me if you have trouble with them, remembering to acknowledge any help you get. Item 5 you should not discuss with anyone else.

1. Let $\mathcal{L}$ be the language with the two place predicate $<$. Define a set of axioms Γ such that $\mathfrak{A}$ is a model for Γ if and only if $\mathfrak{A}$ is a dense linear order without endpoints. Define a set of axioms Σ such that $\mathfrak{A}$ is a model for Σ if and only if $\mathfrak{A}$ is a discrete linear order without endpoints.

2. Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures for $\mathfrak{L}$ then Bob has a winning strategy in $G_\omega(\mathfrak{A}, \mathfrak{B})$ if and only if $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. Use this theorem to prove that any two countable dense linear orderings without endpoints are isomorphic. For each $n = 1, 2, \ldots$ we define a two player game $G_n(\mathfrak{A}, \mathfrak{B})$ between Alice and Bob. The game will have n rounds. On the i^{th} round Alice plays first and either plays $a_i \in |\mathfrak{A}|$ or $b_i \in |\mathfrak{B}|$. On Bob's turn, if Alice played a_i then Bob plays $b_i \in |\mathfrak{B}|$ and if Alice played b_i then Bob plays $a_i \in |\mathfrak{A}|$. The game stops after the n^{th} round and Bob wins if $\{(a_i, b_i) \ : \ i = 1, \ldots, n\}$ is the graph of a partial embedding from $\mathfrak{A}$ into $\mathfrak{B}$. We call $G_n(\mathfrak{A}, \mathfrak{B})$ an Ehrenfeucht-Fraïssé game. You may use the following theorem without proof.

   > **Theorem.** Let $\mathcal{L}$ be a finite language without function symbols and let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$ structures. Then $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ if and only if Bob has a wining strategy in $G_n(\mathfrak{A}, \mathfrak{B})$ for all n.

3. Show that every model of Σ is of the form $(L \times \mathbb{Z}, <)$ where L is a linear order and $<$ is the lexicographic order.

4. Using the above theorem, show that any model $(L \times \mathbb{Z}, <)$ of Σ is elementary equivalent to $(\mathbb{Z}, <)$. Hint: If $x = (a, i)$, $y = (b, j) \in L \times \mathbb{Z}$ we can define a distance function d by $d(x, y) = |i - j|$ if $a = b$ and $d(x, y) = \infty$ if $a \neq b$. The problem for Bob is that Alice can play elements that are infinitely far apart in $L \times \mathbb{Z}$ and force Bob to play elements that are finitely far apart in $\mathbb{Z}$. However, since Bob knows how long the game will last he can play elements sufficiently far apart to avoid conflicts.

5. Discuss the consequences of these results. Is $Cn\,\Sigma$ complete? Decidable? Is Σ countably categorical? What other important conclusions can you come to?

References

[1.] Enderton, Herbert B. A mathematical introduction to logic. Second edition. Harcourt/Academic Press, Burlington, MA, 2001.