Math 69
Winter 2017
Wednesday, January 25

A reminder of some definitions from the reading.

Formally, a structure $\mathfrak{A}$ for a language $\mathcal{L}$ is a function that assigns to each parameter symbol of $\mathcal{L}$ a translation.[1]

The translation of $\forall$ is given by a nonempty set $|\mathfrak{A}|$, the universe of $\mathfrak{A}$. Intuitively, when interpreting formulas of $\mathcal{L}$ in $\mathfrak{A}$, we translate $\forall x$ as "for every element x of $|\mathfrak{A}|$."

If P is a one-place predicate symbol, the translation of P is a subset $P^{\mathfrak{A}}$ of $|\mathfrak{A}|$. Intuitively, $P^{\mathfrak{A}}$ is the set of all elements of the universe for which P is true. If the universe of $\mathfrak{A}$ is $\mathbb{N}$, and we informally translate Px as "x is prime," then formally we would say

$$P^{\mathfrak{A}} = \{n \in \mathbb{N} \mid n \text{ is prime}\}.$$

The same goes for n-place predicate symbols. If we informally translate the 2-place predicate symbol $<$ of the language of arithmetic as "less than," then formally

$$<^{\mathfrak{A}} = \{(n, m) \in \mathbb{N}^2 \mid n < m\}.$$

The translation of a constant symbol c is an element $c^{\mathfrak{A}}$ of $|\mathfrak{A}|$.

The translation of an n-place function symbol f is an n-ary function $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to \mathfrak{A}$.

Definition: A *variable assignment* for $\mathfrak{A}$ is a function s from the set of variables of $\mathcal{L}$ to the universe $|\mathfrak{A}|$.

If s is a variable assignment, x is a variable, and $d \in |\mathfrak{A}|$, the variable assignment $s(x|d)$ agrees with s on all variables other than x and sends x to d. Formally,

$$s(x|d)(y) = \begin{cases} d & \text{if } y = x; \\ s(y) & \text{if } y \neq x. \end{cases}$$

Warning: This is confusing notation: s is a function, but $s(x|d)$ does not mean "s of $x|d$"; rather, $s(x|d)$ is the name of a different function.

[1] A structure is always a structure for a particular language.

Next warning: The textbook uses the notation $\models_{\mathfrak{A}} \varphi[s]$ to mean "the structure $\mathfrak{A}$ with variable assignment s satisfies the formula φ." I am about to use a different notation, $\mathfrak{A} \models \varphi[s]$, both because I am too used to it to change easily, and because it is now pretty universally accepted as standard notation. We'll just have to get used to translating the textbook's notation.

Definition: If s is a variable assignment for $\mathfrak{A}$, we define the extension $\overline{s}$ of s to all terms by recursion; here x denotes any variable symbol, c denotes any constant symbol, f denotes any n-place function symbol, and t_1, t_2,..., t_n any terms.

$$\overline{s}(x) = s(x)$$
$$\overline{s}(c) = c^{\mathfrak{A}}$$
$$\overline{s}(ft_1t_2\cdots t_n) = f^{\mathfrak{A}}(\overline{s}(t_1),\, \overline{s}(t_1),\, \ldots \overline{s}(t_n)).$$

Example: If $\mathcal{L}$ is the language of arithmetic (which the textbook calls the language of elementary number theory), and $\mathfrak{N}$ is the intended model, then, for example,

$$|\mathfrak{N}| = \mathbb{N}$$
$$0^{\mathfrak{N}} = 0$$
$$+^{\mathfrak{N}}(n,m) = n + m$$
$$<^{\mathfrak{N}} = \{(n,m) \in \mathbb{N}^2 \mid n < m\}.$$

If we define a variable assignment s for $\mathfrak{N}$ by $s(v_k) = 2k$, then our definition of $\overline{s}$ tells us

$$\overline{s}(0) = 0$$
$$\overline{s}(v_3) = s(v_3) = 6$$
$$\overline{s}(+v_30) = +^{\mathfrak{N}}(\overline{s}(v_3), \overline{s}(0)) = +^{\mathfrak{N}}(6,0) = 6 + 0 = 6.$$

Definition: We define satisfaction ("truth") of a formula for a given structure $\mathfrak{A}$ and variable assignment s by recursion:

$$\mathfrak{A} \models t_1 = t_2[s] \iff \overline{s}(t_1) = \overline{s}(t_2)$$
$$\mathfrak{A} \models Pt_1 \ldots t_n[s] \iff \langle \overline{s}(t_1), \ldots, \overline{s}(t_n)\rangle \in P^{\mathfrak{A}}$$
$$\mathfrak{A} \models (\neg\alpha)[s] \iff \mathfrak{A} \not\models \alpha[s]$$
$$\mathfrak{A} \models (\alpha \to \beta)[s] \iff \Big[\mathfrak{A} \not\models \alpha[s] \text{ or } \mathfrak{A} \models \beta[s] \text{ (or both)}\Big]$$
$$\mathfrak{A} \models \forall x\, \alpha[s] \iff \text{for every } d \in |\mathfrak{A}| \text{ we have } \mathfrak{A} \models \alpha[s(x|d)].$$

Exercise: Prove carefully and formally, from the definitions, that if $\mathcal{L}$ is a language with a 1-place predicate symbol P, and $\mathfrak{A}$ is a structure for the language $\mathcal{L}$, and s is any variable assignment for $\mathfrak{A}$, then

$$\mathfrak{A} \models \neg\forall x\, Px[s] \iff P^{\mathfrak{A}} \neq |\mathfrak{A}|.$$

Exercise: Show that no one of the following sentences is logically implied by the other two. This is done by giving a structure in which the sentence in question is false, while the other two are true. Your structures can be familiar mathematical structures, familiar non-mathematical structures — for example, $|\mathfrak{A}|$ is the set of all living humans, and $P^{\mathfrak{A}}$ the set of all pairs (a, b) where a is b's mother — or structures that you make up. For this exercise, you can give informal explanations of which sentences are true and which are false in your structures; you need not use the formal definition of satisfaction.

(This is Exercise 2.2.2 in the textbook. You might check out page 82 of the textbook for an example of a finite made-up structure for this language, and a picture describing it.)

(a.) $\forall x \forall y \forall z (Pxy \to (Pyz \to Pxz))$

(b.) $\forall x \forall y (Pxy \to Pyx)$

(c.) $\forall x \exists y Pxy \to \exists y \forall x \; Pxy$

Lemma: If s and r are two variable assignments for $\mathfrak{A}$, and $s(x) = r(x)$ for every variable x that occurs in the term t, then

$$\overline{s}(t) = \overline{r}(t).$$

If $s(x) = r(x)$ for every variable x that occurs *free* in the wff α, then

$$\mathfrak{A} \models \alpha[s] \iff \mathfrak{A} \models \alpha[r].$$

We can prove this lemma by induction on t and on α. You may assume that it is true. You may want to use it in the following exercise.

Exercise: Prove (carefully and formally, from the definitions) the following three related propositions. You may use the lemma on the preceding page.

1. If x does not occur free in α then $\alpha \models \forall x\, \alpha$.
2. In general, $\forall x\, \alpha \models \alpha$ but $\alpha \not\models \forall x\, \alpha$.
3. $\models \alpha$ if and only if $\models \forall x\, \alpha$.

Recall that $\alpha \models \beta$ means that α logically implies β (whenever $\mathfrak{A} \models \alpha[s]$, then $\mathfrak{A} \models \beta[s]$), and $\models \alpha$ means that α is logically valid (for every $\mathfrak{A}$ and s, we have $\mathfrak{A} \models \alpha[s]$). At some point, it might help to note that if s is a variable assignment, x is a variable, and $a \in |\mathfrak{A}|$, then

$$\Big(s(x) = a\Big) \implies \Big(s(x|a) = s\Big).$$

Exercise: Let $\mathcal{L}$ be a language with equality and a 2-place predicate symbol P. For each of the following conditions, find a sentence σ such that the structure $\mathfrak{A}$ is a model of σ (meaning $\mathfrak{A} \models \sigma$, meaning $\mathfrak{A} \models \sigma[s]$ for any variable assignment s) if and only if the condition holds.

Feel free to use all five sentential connectives, use $\exists$, and omit or add parentheses for greatest readability. You do not have to prove formally that your sentence works.

1. $|\mathfrak{A}|$ has exactly two members.

2. $P^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ into $|\mathfrak{A}|$. (That is, each element a of the universe is related to exactly one other element by $P^{\mathfrak{A}}$.)

3. $P^{\mathfrak{A}}$ is a permutation of $|\mathfrak{A}|$. (A permutation is a bijection of $|\mathfrak{A}|$ to itself; that is, a function that is one-to-one and onto).