Math 69
Winter 2017
Friday, January 27

Recall our notation, based on the fact that whether α is satisfied in $\mathfrak{A}$ with s depends only on $\mathfrak{A}$ and $s(x)$ for the *free* variables x occurring in α:

If all the free variables of α are among $v_1, v_2, \ldots, v_n$, then we write

$$\mathfrak{A} \models \alpha[[a_1, a_2, \ldots, a_n]]$$

to mean that $\mathfrak{A}$ satisfies α with any variable assignment s sending each v_i to the corresponding a_i.

Here's some more notation:

If α is a sentence (no variables occur free in α) and $\mathfrak{A}$ a structure, we write

$$\mathfrak{A} \models \alpha$$

to mean that $\mathfrak{A}$ satisfies α with any variable assignment s. We say $\mathfrak{A}$ is a *model* of α. (Informally, we may say α is true in $\mathfrak{A}$.)

If all the free variables of α are among $x, y, \ldots$, we write

$$\mathfrak{A} \models \alpha[(x|a)(y|b)\cdots]$$

to mean that $\mathfrak{A}$ satisfies α with any variable assignment s sending x to a, y to b, ... (This notation is not in the textbook.) This lets us be more freewheeling in our variable names while still being formal:

$$\mathfrak{N} \models x < y \ [(x|2)(y|5)].$$

Definition: If all the free variables of α are among $v_1, v_2, \ldots, v_n$, then in any structure $\mathfrak{A}$, the formula α *defines* a set of n-tuples from $|\mathfrak{A}|^n$, namely

$$\{(a_1, a_2, \ldots, a_n) \mid \mathfrak{A} \models \alpha[[a_1, a_2, \ldots a_n]]\}.$$

Intuitively, the set defined by α is the set of n-tuples for which α is true in $\mathfrak{A}$.

Note that a formula defining a set of n-tuples *must* use only the free variables v_1, v_2, $\ldots$, v_n, although they need not all appear. For example, consider the language of arithmetic, $\mathfrak{L}_{arith}$ (which has equality and symbols $<, 0, S, +, \cdot, E$), and the standard model $\mathfrak{N}$ (which has universe $\mathbb{N}$ and interprets the symbols of $\mathfrak{L}_{arith}$ in the natural way, E denoting exponentiation). If

$$\alpha \text{ is the formula } (\neg\, v_1 < v_3),$$

then α does *not* define a set of pairs. It does define a set of triples:

$$\alpha \text{ defines in } \mathfrak{N} \text{ the set } \{(m, n, p) \mid m \geq p\}.$$

Definition: If α has only the free variable v_1, and the set defined by α in $\mathfrak{A}$ contains only a single element a, then we say α *defines* the element a in $\mathfrak{A}$.

Intuitively, α defines a in $\mathfrak{A}$ if a is the unique element of $|\mathfrak{A}|$ of which α is true.

For example, in $\mathfrak{N}$, the element 2 is defined by the formula $v_1 = SS0$. It is also defined by the formula

$$(\forall x)\,(x < v_1 \;\leftrightarrow\; (x = 0 \vee x = S0)).$$

(1.) Let $\mathfrak{L}$ be the language of first order logic with equality, constant symbols 0 and 1, and two-place function symbols $+$ and $\cdot$, and let

$$\mathfrak{R} = \langle \mathbb{R}, 0, 1, +, \cdot \rangle \,.$$

Give formulas defining each of the following sets or elements of $\mathbb{R}$.

$$\{r \mid r \geq 0\}$$

$$\{(r, s) \mid r \leq s\}$$

$$3$$

$$\frac{1}{2}$$

(2.) Let $\mathfrak{A}$ be a structure for a language $\mathfrak{L}$. Show that the collection of definable subsets of $|\mathfrak{A}|$ always includes $|\mathfrak{A}|$ and $\emptyset$ and is closed under union, intersection, and complement. (By "definable" we mean "definable in the structure $\mathfrak{A}$.")

For a set of sentences Σ in a language $\mathfrak{L}$, we define $Mod\,\Sigma$ to be the class of all structures that are models of Σ For example, if $\mathfrak{L}$ is the language with equality and one two-place predicate symbol P, and Σ is the set containing the three sentences

$$\forall x\, Pxx,$$

$$\forall x\, \forall y\, (Pxy \to (Pyx \to x = y)),$$

$$\forall x\, \forall y\, \forall z\, (Pxy \to (Pyz \to Pxz)),$$

then $Mod\,\Sigma$ is the class of all partial orderings.

If $\Sigma = \{\sigma\}$, instead of $Mod\,\Sigma$ or $Mod\,\{\sigma\}$ we may write $Mod\,\sigma$.

If $\mathfrak{K}$ is a class of structures for a language $\mathfrak{L}$, then $Th\,\mathfrak{K}$ (the *theory* of $\mathfrak{K}$) is the set of all sentences that are satisfied by every structure in $\mathfrak{K}$. For example, if $\mathfrak{K}$ is the class of all equivalence relations, then $Th\,\mathfrak{K}$ is the set of sentences that are true in every equivalence relation.

If $\mathfrak{K} = \{\mathfrak{A}\}$, instead of $Th\,\{\mathfrak{A}\}$ we may write $Th\,\mathfrak{A}$ (the theory of $\mathfrak{A}$).

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ (for the same language) are called elementarily equivalent if they satisfy exactly the same sentences; that is, if $Th\,\mathfrak{A} = Th\,\mathfrak{B}$. We write

$$\mathfrak{A} \equiv \mathfrak{B}.$$

Clearly elementary equivalence is an equivalence relation.

(3.) Show that if Σ is any set of sentences, then:

(a.) If $\mathfrak{A} \in Mod\, \Sigma$ and $\mathfrak{B} \equiv \mathfrak{A}$, then $\mathfrak{B} \in Mod\, \Sigma$.

(We say $Mod\, \Sigma$ is *elementarily closed*.)

(b.) $Th\ Mod\, \Sigma = \{\sigma \mid \sigma \text{ is a sentence and } \Sigma \models \sigma\}$.

(To show two sets of sentences are equal, we generally show that an arbitrary sentence is an element of one set if and only if it is an element of the other.)

(4.) A class $\mathfrak{K}$ of structures is called an *elementary class* (EC) if $\mathfrak{K} = Mod\,\sigma$ for some sentence σ, and an *elementary class in the wider sense* (EC_Δ) if $\mathfrak{K} = Mod\,\Sigma$ for some set of sentences Σ.

Let $\mathfrak{L}$ be the language with equality and a two-place predicate symbol P, and $\mathfrak{K}$ the class of structures for $\mathfrak{L}$ that are equivalence relations.

Show that $\mathfrak{K}$ is EC, by finding a sentence σ such that $\mathfrak{K} = Mod\,\sigma$.

Show that the class of equivalence relations for which each equivalence class has exactly two elements is EC.

Show that the class of equivalence relations for which each equivalence class is infinite is EC_Δ.

(We will be able to show later that this class is *not* EC: There is no way to say each equivalence class is infinite with a single sentence.)

(5.) (Optional, but worth thinking about.) Suppose X is a subset of the structure $\mathfrak{A}$ for the language $\mathfrak{L}$. We can *expand* the language $\mathfrak{L}$ to a larger language $\mathfrak{L}_X$ by adding a new one-place predicate symbol P, and we can *expand* the structure $\mathfrak{A}$ to a new structure $\mathfrak{A}_X$ for the language $\mathfrak{L}_X$ by translating the new symbol P by $P^{\mathfrak{A}_X} = X$. (Informally, we translate Px as "$x \in X$.")

Notice that the expanded structure $\mathfrak{A}_X$ is not larger than $\mathfrak{A}$ in the sense of having more elements, it is expanded in the sense of having "more structure." For example, the field $\langle \mathbb{Q}, 0, 1, +, \cdot \rangle$ is an expansion of the group $\langle \mathbb{Q}, 0, + \rangle$.

If $Y \subseteq |\mathfrak{A}|$ we say that Y is *definable in* $\mathfrak{A}$ *relative to* X (or *from* X) if Y is definable in the expanded structure $\mathfrak{A}_X$. We can write this as $Y \leq_{def} X$.

(a.) Consider the structure $\mathfrak{A} = \langle \mathbb{Z}, \leq \rangle$. The only subsets of $\mathbb{Z}$ that are definable in this structure are $\mathbb{Z}$ itself and $\emptyset$. (We will probably soon see how to show this.) Show that if $X = \{0\}$, every finite or cofinite subset of $\mathbb{Z}$ is definable in $\mathfrak{A}$ relative to X. (A set is *cofinite* if its complement is finite.)

(These are actually the only subsets of $\mathbb{Z}$ that are definable in $\mathfrak{A}$ relative to X. Later we will have the tools we would need to show this.)

(b.) Show (in general, for any language and structure) that $\leq_{def}$ is a preordering on the collection of all subsets of $|\mathfrak{A}|$.

(c.) As in the mathematical structures handout, because $\leq_{def}$ is a preordering we know we can define an equivalence relation by

$$X \equiv Y \iff X \leq_{def} Y \;\&\; Y \leq_{def} X.$$

(We say that X and Y are equidefinable.) Then we can define a partial ordering of equivalence classes by

$$[X] \leq_{def} [Y] \iff X \leq_{def} Y.$$

What can you say about the structure of this partial ordering?

You can think about this question in general, or in a specific case, say the case of the structure $\mathfrak{N} = \langle \mathbb{N}, 0, S, +, \cdot, E, \leq \rangle$. We can actually say *much* more in this specific case than we can in general.

A specific question: Can you produce a language $\mathcal{L}$, a structure $\mathfrak{A}$ for $\mathcal{L}$, and two subsets $X \subset |\mathfrak{A}|$ and $Y \subset |\mathfrak{A}|$, such that neither X nor Y is definable in the other? (That is, can you show that the partial ordering $\leq_{def}$ on equivalence classes is not always linear?)