Math 69
Winter 2017
Monday, January 30

(1.) Let h be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and $s : V \to |\mathfrak{A}|$ (where V is the set of variable symbols) be a variable assignment for $\mathfrak{A}$. The composition $h \circ s$ defines a variable assignment for $\mathfrak{B}$.

Use induction on terms to show that, for every term t, we have

$$h(\overline{s}(t)) = \overline{h \circ s}(t).$$

(2.) Let $\mathfrak{A}$ be the structure $\langle \mathbb{Q}, < \rangle$.

For any rational numbers $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$, there is an automorphism of $\mathfrak{A}$ taking every p_i to q_i. You may use this fact.

Which subsets of $\mathbb{Q}$ are definable in $\mathfrak{A}$? Show your answer is correct.

Which subsets of $\mathbb{Q} \times \mathbb{Q}$ are definable in $\mathfrak{A}$? (There are exactly eight of them.) You do not have to prove your answer is correct, but you should have some idea of how to do so.

(3.) Let $\mathcal{L}$ be the language for first-order logic with two-place predicate symbols E and P and one-place function symbol f, and no other predicate or function symbols.

Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}$ such that $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$, the relation $P^{\mathfrak{A}}$ induces a well-defined relation on equivalence classes, and the function $f^{\mathfrak{A}}$ induces a well-defined function on equivalence classes.

We can define a new structure $\mathfrak{B}$ whose elements are the equivalence classes of $\mathfrak{A}$, in which E is translated as equality, and P and f are translated as the relation and function induced by $P^{\mathfrak{A}}$ and $f^{\mathfrak{A}}$.

Show that the function h from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ sending a to $[a]$ (the equivalence class of a) is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Using the homomorphism theorem, what can we conclude about $Th(\mathfrak{A})$ and $Th(\mathfrak{B})$? Remember that the language does not have the equality symbol.

(4.) If you get this far, prove directly (without using of the homomorphism theorem) that if $\mathfrak{A}$, $\mathfrak{B}$, and h are as in problem (3), and s is a variable assignment for $\mathfrak{A}$, then for every wff α,

$$\big(\mathfrak{A} \models \alpha[s]\big) \iff \big(\mathfrak{B} \models \alpha[h \circ s]\big).$$

Use induction on α. Problem (1) will help with the base case. Remember that the language does not have the equality symbol.