Math 69
Winter 2017
Tuesday, January 31
Extra Problems

Two old definitions and two new ones:

Definition: A variable assignment s extends to a function $\overline{s}$ on all terms defined recursively by:

$$\overline{s}(x) = s(x) \text{ for a variable symbol x;}$$

$$\overline{s}(c) = c^{\mathfrak{A}} \text{ for a constant symbol c;}$$

$$\overline{s}(ft_1t_2\cdots t_n) = f^{\mathfrak{A}}(\overline{s}(t_1), \overline{s}(t_2), \ldots, \overline{s}(t_n)).$$

Definition: If s is a variable assignment for $\mathfrak{A}$ and $d \in |\mathfrak{A}j|$ then we define a new variable assignment $s(xjd)$ by

$$s(xjd)(v) = \begin{cases} d & \text{if } v = x; \\ s(v) & \text{if } v \neq x. \end{cases}$$

That is, we change the translation s so that x is translated as d.

Definition: If t and u are terms and x is a variable, we define t_u^x to be the term obtained from t by replacing all occurrences of x with u.

Definition: If α is a wff, we define α_u^x to be the wff obtained from α by replacing every free occurrence of the variable symbol x with the term u.

For example, suppose u is $SS0$.

If t is $+xy$ then t_u^x is $+SS0y$.

If α is $\exists y(x = y + y)$ then α_u^x is $\exists y(SS0 = y + y)$.

If α is $\exists x(S0 = x)$ then α_u^x is $\exists x(S0 = x)$.

Exercise 1: Give a definition of t^x_u by recursion on t.

Exercise 2: Prove that if $\overline{s}(u) = d$ then

$$\overline{s}(t_u^x) = \overline{s(x|d)}(t) :$$

Intuitively, if the term u is translated as d, then replacing the variable x in t with the term u has the same effect (on the translation of t) as translating x as d. Use your recursive definition of t_u^x.

Exercise 3: Give a definition of α_u^x by recursion on α. Be careful with the $\forall v$ step; the cases $v = x$ and $v \neq x$ are different. Remember that we have already given a definition of t_u^x where t is any term.

Exercise 4: Prove that for any structure $\mathfrak{A}$, variable assignment s, variable symbol x, term u, and quantifier-free formula α (that is, the symbol $\forall$ does not occur in α), we have

$$\mathfrak{A} \models \alpha[s(x|\overline{s}(u))] \iff \mathfrak{A} \models \alpha^x_u[s].$$

Use your recursive definition and the result of exercise 2:

$$\overline{s}(t^x_u) = \overline{s(x|\overline{s}(u))}(t).$$

Exercise 5: Suppose that α is a formula, x a variable symbol, and u a term such that (*) holds:

(*) For any structure $\mathfrak{A}$ and any variable assignment s, we have

$$\mathfrak{A} \models \alpha[s(x|\overline{s}(u))] \iff \mathfrak{A} \models \alpha^x_u[s].$$

Show that for any structure $\mathfrak{A}$ and any variable assignment s we have

$$\mathfrak{A} \models (\forall x\alpha \to \alpha^x_u)[s].$$

Exercise 4 shows that if α is quantifier-free, then (*) holds.
Conclude that if α is quantifier free, then $(\forall x\alpha \to \alpha^x_u)$ is logically valid.)

We might try to prove the false claim that for any α, x, u, and $\mathfrak{A}$, we have

$$\mathfrak{A} \models (\forall x \alpha \to \alpha^x_u)[s]$$

(that is, that $(\forall x \alpha \to \alpha^x_u)$ is logically valid) by proving the equally false claim that (*) holds for all α, x, and u.

(The original claim is certainly false. For example, if α is $\exists y(y \neq x)$ and u is y, then $(\forall x \alpha \to \alpha^x_u)$ is

$$\forall x \exists y \, y \neq x \ \to \exists y \, y \neq y,$$

which is certainly not logically valid.)

(*) does hold for all quantifier-free formulas, which you showed by induction in exercise (4). The only missing step in the inductive proof that (*) always holds is the $\forall$ step.

Exercise 6: Assume as inductive hypothesis that for any structure $\mathfrak{A}$ and any variable assignment s, we have

$$\mathfrak{A} \models \alpha[s(x|\overline{s}(u))] \iff \mathfrak{A} \models \alpha^x_u[s].$$

Try to show that if $\beta = \forall v \alpha$, then for any structure $\mathfrak{A}$ and any variable assignment s, we have

$$\mathfrak{A} \models \beta[s(x|\overline{s}(u))] \iff \mathfrak{A} \models \beta^x_u[s].$$

What goes wrong? Are there some special cases in which you can succeed in proving this? How far can you push it?