Math 69
Winter 2017
Wednesday, January 4

Propositional Logic and Truth Tables

Here are some propositional connectives.

Not: $\neg$
And: $\wedge$
Or: $\vee$
If ... then: $\rightarrow$
If and only if: $\leftrightarrow$

One way to give the meaning of a propositional connective is via a *truth table*, such as the following truth table for $\wedge$, which gives the truth value of $(P \wedge Q)$ ("P and Q") depending on the truth values of P and Q:

P	Q	$(P \wedge Q)$
T	T	T
T	F	F
F	T	F
F	F	F

Try writing down truth tables for $\vee$ and for $\rightarrow$, where $(P \vee Q)$ means "P or Q" and $(P \rightarrow Q)$ means "if P then Q."

P	Q	$(P \vee Q)$
T	T	
T	F	
F	T	
F	F	

P	Q	$(P \rightarrow Q)$
T	T	
T	F	
F	T	
F	F	

A second way to give the meaning of a propositional connective is via a *Boolean function*, such as the following Boolean function $Val_\wedge$ for $\wedge$, which takes the truth values of P and Q as arguments (inputs), and gives the truth value of $(P \wedge Q)$ as value (output):

$$Val_\wedge(T, T) = T$$
$$Val_\wedge(T, F) = Val_\wedge(F, T) = Val_\wedge(F, F) = F.$$

Another way to phrase the same definition is:

$$Val_\wedge(x, y) = \begin{cases} T & \text{if } x = y = T; \\ F & \text{otherwise.} \end{cases}$$

Try writing down Boolean functions for $\neg$ and $\leftrightarrow$, where $(\neg P)$ means "not P" and $(P \leftrightarrow Q)$ means "P if and only if Q."

We can define a connective to be a Boolean function. How many possible binary (two-place) connectives are there:

— If we include all two-place Boolean functions?

— If we include only those two-place Boolean functions whose value depends on both arguments? For example, we would not include the Boolean function G given by

$$G(T,T) = G(T,F) = T$$
$$G(F,T) = G(F,F) = F,$$

as its value depends only on the first argument.

On page 51 of the textbook, you can find a list of all binary connectives, with standard symbols for each.

We can translate some English sentences into the language of sentential logic. Begin by assigning sentence symbols to the simple declarative statements that are combined to produce these more complex sentences. This sometimes calls for exercising some judgment; the relevant simple declarative sentences may not appear word-for-word as parts of the English sentence, and the English may be ambiguous. For example:

English sentence: The exam will be on Monday or Tuesday.

Sentence components:
M: The exam will be on Monday.
T: The exam will be on Tuesday.

Sentence translation:
$((M \vee T) \wedge (\neg(M \wedge T)))$

Here I have assumed from context that the English "or" is to be interpreted as exclusive. Since the sentential logic "$\vee$" is inclusive, I had to use a translation into sentential logic that does not exactly mirror the syntax of the English sentence.

An alternative translation, equivalent as far as truth value goes (see if you can convince yourself of this) but even farther from the original English syntax, is:
$(M \leftrightarrow (\neg T))$
Another, perhaps closer to the original English syntax, is:
$((M \wedge (\neg T)) \vee (T \wedge (\neg M)))$

On the next page are sentences for you to translate. All use the same simple component sentences, to which we can assign the indicated sentence symbols:

"x is prime.": P

"x is even.": E

"x equals 2.": D

You should use a pair of parentheses for every connective symbol you use: write "$(\neg P)$" rather than "P", and "$(P \vee (Q \wedge R))$" rather than "$P \vee (Q \wedge R)$" or (significantly worse) "$P \vee Q \wedge R$". (Can you see why "$P \vee Q \wedge R$" is problematic?)

(1,) x is even but x is not equal to 2.

(2.) If x is prime then x is not even.

(3.) x is not even if x is prime.

(4.) x is prime only if x is not even or x is equal to 2.

(5.) x is an even prime exactly in case x is equal to 2.

A language $\mathcal{L}$ for propositional logic (sentential logic) is defined as follows:

The *symbols* of the language are the following:
Infinitely many propositional symbols (sentence symbols) A_1, A_2, A_3, $\ldots$,
Connective symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$
Punctuation symbols (and).

Any finite sequence of symbols is an *expression*.

The *(well formed) formulas*, or *wff's*, of the language are the expressions formed according to the following rules:

1. Any propositional symbol is a wff.
2. If α is a wff, so is $(\neg\alpha)$.
3. If α and β are wffs, so is $(\alpha * \beta)$, where $*$ is any one of the binary connective symbols, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

Intuitively, the well formed formulas are the meaningful expressions of the language. However, this is a formal definition and so it is quite rigid. If A, B and C are sentence symbols, the expression

$$(A \wedge B \wedge C)$$

is not a well formed formula. However, the following two expressions are wffs:

$$((A \wedge B) \wedge C);$$

$$(A \wedge (B \wedge C)).$$

It is clear what we mean by $(A \wedge B \wedge C)$, and quite soon we will adopt "parenthesis omitting" conventions by which this expression is considered a (meaningful) abbreviation for $(A \wedge (B \wedge C))$.

The text adds an additional rule to the three given above: "No expression is a wff unless it is compelled to be one by (1), (2) and (3)."

This can be made precise by adopting either of two different formal definitions for the set of well formed formulas.

Definition A. An expression α is a (well formed) formula iff there is a finite sequence of expressions $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that, for each i, one of:

1. α_i is a sentence symbol;
2. $\alpha_i = (\neg\alpha_j)$ for some $j < i$;
3. $\alpha_i = (\alpha_j * \alpha_k)$ for some $i, j < k$, where $*$ is any one of the binary connective symbols, $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$;

and $\alpha_n = \alpha$.
We may call this a *construction sequence* for α.

Definition B. An expression α is a (well formed) formula iff α is in every set X of expressions such that:

1. every sentence symbol is in X.;
2. whenever β is in X, then $(\neg\beta)$ is in X;
3. whenever β and γ are in X, then $(\beta * \gamma)$ is in X, where $*$ is any one of the binary connective symbols, $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.

Such a set X contains all the sentence symbols and is *closed* under the formula-building operations.

How could we show that A and B define exactly the same collections of well formed formulas? (For any expression α, we must show that α is a wff according to definition A if and only if α is a wff according to definition B.)

Finally, for future reference, here is the inductive method for proving that some proposition $\varphi(\alpha)$ holds for all wff's α:

1. (Base Step) Prove that $\varphi(A)$ holds for every sentence symbol A.
2. (Inductive Step for $\neg$) Prove that if $\varphi(\alpha)$ holds then $\varphi((\neg\alpha))$ holds.
3. (Inductive Step for $\wedge$) Prove that if $\varphi(\alpha)$ and $\varphi(\beta)$ hold, then $\varphi((\alpha\wedge\beta))$ holds.
4. (Inductive Step for $\vee$) Prove that if $\varphi(\alpha)$ and $\varphi(\beta)$ hold, then $\varphi((\alpha\vee\beta))$ holds.
5. (Inductive Step for $\rightarrow$) Prove that if $\varphi(\alpha)$ and $\varphi(\beta)$ hold, then $\varphi((\alpha \rightarrow \beta))$ holds.
6. (Inductive Step for $\leftrightarrow$) Prove that if $\varphi(\alpha)$ and $\varphi(\beta)$ hold, then $\varphi((\alpha \leftrightarrow \beta))$ holds.

For some purposes, the last four inductive steps can be combined into a single inductive step for binary connectives: Prove that if $\varphi(\alpha)$ and $\varphi(\beta)$ hold, then $\varphi((\alpha * \beta))$ holds, where $*$ is any binary connective.

Example: Prove that no wff has length 2.[1]

Proof: We show by induction that the length of every wff α is either equal to 1 or greater than or equal to 4.[2]

Base Step: If α is a sentence symbol A, then the length of α is 1.

Inductive Step for $\neg$: If α has length n, where $n = 1$ or $n \geq 4$, then $(\neg\alpha)$ has length $n + 3 \geq 4$.

Inductive Step for Binary Connectives: If α has length n, where $n = 1$ or $n \geq 4$, and β has length m, where $m = 1$ or $m \geq 4$, then if $*$ is any binary connective, $(\alpha * \beta)$ has length $m + n + 3 \geq 5$.

This completes the proof.

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[1]The length of a wff is the number of symbols in it. See exercise 2 of Section 1.1; this proof can be an ingredient in the solution.

[2]Our proposition $\varphi(\alpha)$ is "The length of α is either equal to 1 or greater than or equal to 4." We are proving this holds for every wff α.

Note, the point of a proof by this inductive method is to show that if we define a set of expressions X by

$$X = \{\alpha \mid \varphi(\alpha)\},$$

then X contains all the sentence symbols and is closed under the formula-building operations, and therefore, by Definition B, every wff is in X. Hence, $\varphi(\alpha)$ holds for every wff α.

Alternatively, it follows from a proof by this method that if $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a construction sequence as in Definition A, then, by induction on i, $\varphi(\alpha_i)$ holds for every $i \leq n$. Hence, $\varphi(\alpha)$ holds for every wff α.

Example: Prove by induction that if α is any wff, then the number of left parentheses in α, the number of right parentheses in α, and the number of connective symbols in α are equal.