Math 69
Winter 2013
Monday, January 14

Compactness

Here are some problems from the last class; some have solutions included. Continue with the ones you did not get to last time, and make sure you understand the solutions of the others. Then go on to the new problems.

Problem: Prove the following proposition, which will be useful later in this handout, by induction.

Proposition: For every two truth assignments v and w that agree with each other on every sentence symbol that occurs in α, we have $\overline{v}(\alpha) = \overline{w}(\alpha)$.

Example Solution: We prove the proposition by induction on formulas.

Base Case: If α is a sentence symbol A_i, by assumption $v(A_i) = w(A_i)$. By the definition of truth assignment, $\overline{v}(A_i) = v(A_i)$ and $\overline{w}(A_i) = w(A_i)$. Therefore

$$\overline{v}(\alpha) = \overline{v}(A_i) = v(A_i) = w(A_i) = \overline{w}(A_i) = \overline{w}(\alpha).$$

Inductive Step for Negation: Suppose that $\beta = (\neg\alpha)$, and the theorem holds for α; we must show the theorem holds for β. If v and w do not agree on all the sentence symbols in β, there is nothing to prove. If they do, then they also agree on all the sentence symbols in α, and by inductive hypothesis[1] $\overline{v}(\alpha) = \overline{w}(\alpha)$. Therefore, using the definition of truth assignment,

$$\overline{v}(\beta) = \overline{v}((\neg\alpha)) = Val_\neg(\overline{v}(\alpha)) = Val_\neg(\overline{w}(\alpha)) = \overline{w}((\neg\alpha)) = \overline{w}(\beta).$$

Inductive Step for Binary Connectives: Suppose that $\beta = (\alpha * \gamma)$, where $*$ is any binary connective, and the theorem holds for α and γ; we must show the theorem holds for β. If v and w do not agree on all the sentence symbols in β, there is nothing to prove. If they do, then they also agree on all the sentence symbols in α and γ, and by inductive hypothesis $\overline{v}(\alpha) = \overline{w}(\alpha)$ and $\overline{v}(\gamma) = \overline{w}(\gamma)$. Therefore, using the definition of truth assignment,

$$\overline{v}(\beta) = \overline{v}((\alpha*\gamma)) = Val_*(\overline{v}(\alpha), \overline{v}(\gamma)) = Val_*(\overline{w}(\alpha), \overline{w}(\gamma)) = \overline{w}((\alpha*\gamma)) = \overline{w}(\beta).$$

[1] The inductive hypothesis is the assumption that the theorem holds for α.

Problem: Show that the following are tautologically equivalent:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta$$

$$\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))$$

Example Solution: We must show that for any truth assignments v,

$$\overline{v}((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta) = \overline{v}(\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))).$$

First, suppose $\overline{v}((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta) = T$. This can happen in one of two ways.

Case 1: $\overline{v}(\beta) = T$. In this case we have $\overline{v}(\alpha_1 \to \beta) = T$, from which it follows that $\overline{v}(\alpha_2 \to (\alpha_1 \to \beta)) = T$, and so forth, until we get to $\overline{v}(\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))) = T$. (To be formal, we would argue by induction on n.)

Case 2: $\overline{v}(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) = F$. In this case, we have $\overline{v}(\alpha_i) = F$ for some i. It follows that $\overline{v}(\alpha_i \to (\alpha_{i-1} \to (\cdots (\alpha_1 \to \beta) \cdots))) = T$, from which it follows that $\overline{v}(\alpha_{i+1} \to (\alpha_i \to (\alpha_{i-1} \to (\cdots (\alpha_1 \to \beta) \cdots)))) = T$, and so forth, until we get to $\overline{v}(\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))) = T$.

Second, suppose $\overline{v}((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta) = F$. This can happen in only one way; $\overline{v}(\alpha_1 \wedge \alpha_2 \wedge \cdots \alpha_n) = T$ and $\overline{v}(\beta) = F$. Since $\overline{v}(\beta) = F$ and $\overline{v}(\alpha_1) = T$, we have $\overline{v}(alpha_1 \to \beta) = F$. Since also $\overline{v}(\alpha_2) = T$, it follows that $\overline{v}(\alpha_2 \to (\alpha_1 \to \beta)) = F$, and so forth, until we get to $\overline{v}(\alpha_n \to (\alpha_{n-1} \to (\cdots (\alpha_1 \to \beta) \cdots))) = F$.

Note: Only at the end of the proof is it fair to say the two formulas are tautologically equivalent. For example, to say at the end of Case 1 above, "so in this case, the formulas are tautologically equivalent," would be wrong. Tautologically equivalent means *every* truth assignment gives them the same value, and this has not yet been proven.

For example, suppose you were proving, "Every number is a jabberwock." If you first assume n is even and prove n is a jabberwocky, you can't then say, "So in the case that n is even, every number is a jabberwock."

Problem: Show the following: $\Sigma \models \alpha$ if and only if $\Sigma \cup \{\neg\alpha\}$ is not satisfiable.

Example Solution: First suppose $\Sigma \models \alpha$; we must show that $\Sigma \cup \{\neg\alpha\}$ is not satisfiable.

To do this, let v be an arbitrary truth assignment, and show that v does not satisfy $\Sigma \cup \{\neg\alpha\}$.

Case 1: The truth assignment v does not satisfy Σ. Then, since Σ is contained in $\Sigma \cup \{\neg\alpha\}$, it follows that v does not satisfy $\Sigma \cup \{\neg\alpha\}$.

Case 2: The truth assignment v does satisfy Σ. Since $\Sigma \models \alpha$, it follows that v satisfies α, so $\overline{v}(\alpha) = T$ and $\overline{v}(\neg\alpha) = F$. That is, v does not satisfy $\neg\alpha$, so it follows that v does not satisfy $\Sigma \cup \{\neg\alpha\}$.

Since this covers all the cases, $\Sigma \cup \{\neg\alpha\}$ is not satisfiable.

Second, suppose that $\Sigma \cup \{\neg\alpha\}$ is not satisfiable, and show that $\Sigma \models \alpha$.

To do this, let v be a truth assignment that satisfies Σ, and show v satisfies α; that is, that $\overline{v}(\alpha) = T$.

Suppose not. Then $\overline{v}(\alpha) = F$, and so $\overline{v}(\neg\alpha) = T$. But since v satisfies everything in Σ, and v also satisfies $\neg\alpha$, it follows that v satisfies $\Sigma \cup \{\neg\alpha\}$. This is a contradiction, since $\Sigma \cup \{\neg\alpha\}$ is not satisfiable. Therefore, v satisfies α.

Note: Notice that in writing up the proof, we analyze the logic of the statement, and appeal to definitions. Here we want to prove $A \iff B$, so we prove $A \implies B$ and $B \implies A$. To prove $A \implies B$, we assume A and prove B.

A is $\Sigma \models \alpha$, which we have defined to mean "Every truth valuation that satisfies Σ also satisfies α," so to prove it, we begin by letting v be an arbitrary truth assignment that satisfies Σ.

Another Note: Our formal language model represents some of this mathematical activity. To prove $A \iff B$, we prove $A \implies B$ and $B \implies A$. This is reflected in our formal system by the fact that

$$\{(A \implies B),\ (B \implies A)\} \models (A \iff B).$$

To prove "Every truth valuation that satisfies Σ also satisfies α," we let v be a name for an arbitrary truth valuation that satisfies Σ, and prove that v also satisfies α. Our formal system is not strong enough to capture this, because "for all" is not one of the logical concepts we have built into our language. The language of first-order logic, or predicate logic, which we will see next week, does capture this.

Problem: Show the following:

If Σ is satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is satisfiable.

A set of formulas Σ is said to be *finitely satisfiable* if every finite subset of Σ is satisfiable. We are about to prove the Compactness Theorem: If Σ is finitely satisfiable, then Σ is satisfiable.

Problem: Prove the following proposition, which we will use as a lemma:

Proposition: If Σ is finitely satisfiable, then at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ is finitely satisfiable.

Hint: Suppose not. Since $\Sigma \cup \{\alpha\}$ is not finite satisfiable, there is a finite subset $\Sigma_0 \subset \Sigma$ such that $\Sigma_0 \cup \{\alpha\}$ is not satisfiable. Similarly, there is a finite subset $\Sigma_1 \subset \Sigma$ such that $\Sigma_1 \cup \{\neg\alpha\}$ is not satisfiable. Deduce a contradiction by finding a finite subset of Σ that is not satisfiable.

Problem: Here is an outline of the proof of the Compactness Theorem. Fill in the missing details.

Suppose that Σ is finitely satisfiable. We must show that Σ is satisfiable.

Define, by induction on n,

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\} & \text{if this is finitely satisfiable;} \\ \Sigma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Show that each Σ_n is finitely satisfiable.

Hint: Use induction on n, and the previous proposition.

Now let $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$. Show that Σ^* is finitely satisfiable.

Hint: Argue that any finite subset of Σ^* must already be contained in one of the Σ_n.

Note that $\Sigma \subseteq \Sigma^*$, and that for each n, either A_n or $\neg A_n$ is in Σ^* (but not both). Define a truth assignment v by

$$v(A_n) = \begin{cases} T & A_n \in \Sigma^* \\ F & \neg A_n \in \Sigma^* \end{cases}$$

Show that v satisfies Σ^*, and therefore Σ (showing that Σ is satisfiable), as follows:

Suppose not. Let $\alpha \in \Sigma^*$ with $\overline{v}(\alpha) = F$. For each sentence symbol A_n, define

$$\beta_n = \begin{cases} A_n & A_n \in \Sigma^* \\ \neg A_n & \neg A_n \in \Sigma^* \end{cases}$$

Let Γ be the finite subset of Σ^* defined by

$$\Gamma = \{\alpha\} \cup \{\beta_n \mid A_n \text{ occurs in } \alpha\}.$$

Because Σ^* is finitely satisfiable, there is a truth assignment w satisfying Γ. Deduce a contradiction.

Hint: Show that v and w agree on every sentence symbol contained in α, and use the proposition on page 1 to deduce a contradiction.

Deductions, Soundness and Completeness

If Σ is a set of formulas and α is a formula, we define a *deduction* of α from Σ to be a finite sequence of formulas

$$\begin{array}{l}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n\end{array}$$

whose last element α_n is α, such that for each $i \leq n$, one of the following conditions holds:

1. The formula α_i is a tautology (a *logical axiom.*)
2. The formula α_i is a member of Σ (a *hypothesis.*)
3. There are j and k less than i such that

   $$\alpha_j = \beta$$

   $$\alpha_k = (\beta \to \gamma)$$

   $$\alpha_i = \gamma.$$

   (The formula α_i is derived from α_j and α_k by *modus ponens.*)

If there is a deduction of α from Σ, we write $\Sigma \vdash \alpha$. For example, $\{(A \wedge B)\} \vdash A$, as shown by the following deduction:

(1.) $(A \wedge B)$

(2.) $((A \wedge B) \to A)$

(3.) A

Explain why each line is legitimate according to the definition of deduction.

According to our definitions, "Every truth valuation satisfying Σ also satisfies α" is true in case there are no truth valuations that satisfy Σ. That is, if Σ is not satisfiable, then $\Sigma \models \alpha$. For instance,

$$\{A,\, (\neg A)\} \models B.$$

Show that also

$$\{A,\, (\neg A)\} \vdash B.$$

Soundness Theorem: If $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$.

This says that our notion of deduction is sound: We cannot deduce α from Σ unless α actually does (tautologically) follow from Σ. So, giving a deduction that demonstrates $\Sigma \vdash \alpha$ actually proves $\Sigma \models \alpha$.

Prove the Soundness Theorem. Suggestion: Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction of α from Σ, and prove by (strong) induction on i that $\Sigma \models \alpha_i$ for each $i \leq n$.

(In strong induction, to prove $\Sigma \models \alpha_i$, instead of assuming only that $\Sigma \models \alpha_{i-1}$, you assume that *for all* $j < i$, $\Sigma \models \alpha_j$.)

You might or might not want to separately prove as a lemma that, for any set of wffs Σ and any wffs β and γ, if $\Sigma \models \beta$ and $\Sigma \models (\beta \to \gamma)$, then $\Sigma \models \gamma$.

Completeness Theorem: If $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$.

This is the converse of the Soundness Theorem. It says that our notion of deduction is complete: If α tautologically follows from Σ, then we can deduce α from Σ.

That is, given a set Σ of wffs and a wff α, either there is a deduction of α from Σ, or there is a translation that makes every wff in Σ true but makes α false.

Prove the Completeness Theorem.

Suggestion: Use the Compactness Theorem to prove that if $\Sigma \models \alpha$ then there is a finite $\Gamma \subset \Sigma$ such that $\Gamma \models \alpha$.

You might also want to prove separately as a lemma that

$$(\alpha_n \rightarrow (\alpha_{n-1} \rightarrow (\cdots \alpha_1 \rightarrow \beta) \cdots))$$

is a tautology if and only if

$$\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1\} \models \beta.$$

It may help that you have already showed that $\Sigma \models \alpha$ iff $\Sigma \cup \{(\neg\alpha)\}$ is not satisfiable, and that

$$(\alpha_n \rightarrow (\alpha_{n-1} \rightarrow (\cdots \alpha_1 \rightarrow \beta) \cdots)) \models\!\!\!= \!\!| \ ((\alpha_n \wedge \alpha_{n-1} \wedge \cdots \wedge \alpha_1) \rightarrow \beta).$$

More space for your proof:

Preview of a homework problem from the textbook: In 1977 it was proved that every planar map can be colored with four colors. Of course, the definition of "map" requires that there be only finitely many countries. But extending the concept, suppose we have an infinite (but countable) planar map with countries $C_1, C_2, C_3, \ldots$. Prove that this infinite planar map can still be colored with four colors.

Suggestion: Use four infinite (countable) collections of sentence symbols, one for each color. One sentence symbol, for example, can be used to mean "Country C_7 is colored red." Form a set Σ_1 of wffs that say each country is colored exactly one color. For example, one sentence in Σ_1 will say that C_7 is colored exactly one color. Form another set Σ_2 of wffs that say, for each pair of adjacent countries, that they are not the same color. For example, if C_7 is adjacent to C_2, then one sentence in Σ_2 will say that C_2 and C_7 are not colored the same color. Apply compactness to $\Sigma_1 \cup \Sigma_2$.

Note: The intention here is that you are given a *specific* infinite map, and you use this map to produce your sets of wffs. That is, for each infinite countable planar map $\mathcal{M}$, there is a set $\Sigma_{\mathcal{M}}$ of wffs, such that applying compactness to $\Sigma_{\mathcal{M}}$ demonstrates that $\mathcal{M}$ can be colored with four colors.