Math 69
Winter 2017
Friday, January 13

Here are two things from earlier handouts you may not have gotten to, plus a preview of a homework problem.

Completeness Theorem: If $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$.

This is the converse of the Soundness Theorem. It says that our notion of deduction is complete: If α tautologically follows from Σ, then we can deduce α from Σ.

That is, given a set Σ of wffs and a wff α, either there is a deduction of α from Σ, or there is a translation [a truth valuation] that makes every wff in Σ true but makes α false.

Prove the Completeness Theorem.

Suggestion: Use the Compactness Theorem to prove that if $\Sigma \models \alpha$ then there is a finite $\Gamma \subset \Sigma$ such that $\Gamma \models \alpha$.

You might also want to prove separately as a lemma that

$$(\alpha_n \to (\alpha_{n-1} \to (\cdots \alpha_1 \to \beta) \cdots))$$

is a tautology if and only if

$$\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_1\} \models \beta.$$

It may help that you have already showed that $\Sigma \models \alpha$ iff $\Sigma \cup \{(\neg\alpha)\}$ is not satisfiable, and that

$$(\alpha_n \to (\alpha_{n-1} \to (\cdots \alpha_1 \to \beta) \cdots)) \models\dashv ((\alpha_n \wedge \alpha_{n-1} \wedge \cdots \wedge \alpha_1) \to \beta).$$

More space for your proof:

Effectiveness

We have already seen that:

$\{\alpha \mid \alpha \text{ is a wff}\}$ is decidable. (The parsing algorithm of Section 1.3 determines whether an expression is a wff.)

$\{\alpha \mid \alpha \text{ is a tautology}\}$ is decidable. (The method of truth tables is effective.)

$\{(\alpha, \beta) \mid \alpha \models \beta\}$ is decidable.

If Σ is decidable, so is $\{\vec{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_n) \mid \vec{\alpha} \text{ is a deduction from } \Sigma\}$.

Every decidable set is also effectively enumerable. (There is an effective procedure to list *all* expressions, or pairs of expressions, or finite sequences of expressions. To list a decidable set of expressions X, run the procedure to list the set E of all expressions; after listing an expression in E, test whether it is in X using an effective decision procedure for X, and if it is, then list it in X.)

A set X is effectively enumerable iff it is semidecidable.

A nonempty set X is effectively enumerable iff it is the range of a computable function on $\mathbb{N}$.

If Σ is a decidable set of wffs, $\{\alpha \mid \Sigma \vdash \alpha\}$ is effectively enumerable.

If Σ is a decidable set of wffs, $\{\alpha \mid \Sigma \models \alpha\}$ is effectively enumerable.

Suppose that Σ is a decidable set of wffs and for every wff α we have $\Sigma \models \alpha$ or $\Sigma \models (\neg\alpha)$, but not both. Then $\{\alpha \mid \Sigma \models \alpha\}$ is decidable.

Do the following problem:

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function whose range is not decidable. (Take it on faith for now that such a function exists.) Let Σ be the set of wffs of the form $\neg\neg\neg\neg \cdots A_{f(n)}$, where $A_{f(n)}$ is preceded by n-many double negations. Show that Σ is decidable but $\{\alpha \mid \Sigma \vdash \alpha\}$ is not decidable. (Feel free to use Soundness and Completeness.)

Preview of a homework problem from the textbook: In 1977 it was proved that every planar map can be colored with four colors. Of course, the definition of "map" requires that there be only finitely many countries. But extending the concept, suppose we have an infinite (but countable) planar map with countries $C_1, C_2, C_3, \ldots$. Prove that this infinite planar map can still be colored with four colors.

Suggestion: Use four infinite (countable) collections of sentence symbols, one for each color. One sentence symbol, for example, can be used to mean "Country C_7 is colored red." Form a set Σ_1 of wffs that say each country is colored exactly one color. For example, one sentence in Σ_1 will say that C_7 is colored exactly one color. Form another set Σ_2 of wffs that say, for each pair of adjacent countries, that they are not the same color. For example, if C_7 is adjacent to C_2, then one sentence in Σ_2 will say that C_2 and C_7 are not colored the same color. Apply compactness to $\Sigma_1 \cup \Sigma_2$.

Note: The intention here is that you are given a *specific* infinite map, and you use this map to produce your sets of wffs. That is, for each infinite countable planar map $\mathcal{M}$, there is a set $\Sigma_{\mathcal{M}}$ of wffs, such that applying compactness to $\Sigma_{\mathcal{M}}$ demonstrates that $\mathcal{M}$ can be colored with four colors.