Mathematics 69
Winter 2017
Homework Problem Assigned Monday, January 30

Let $\mathcal{L}$ be the language for first-order logic with two-place predicate symbols E and P and one-place function symbol f. (We are not assuming that $\mathcal{L}$ has the equality symbol. On the other hand, we are not ruling out the possibility that $\mathfrak{L}$ has the equality symbol and/or any number of parameter symbols in addition to $\forall$, E, P, and f. Other symbols are not relevant to this question.)

Suppose $\mathfrak{A}$ is a structure for $\mathcal{L}$ that is a model of the sentence

$$\forall x E x x$$

and of every sentence of the form

$$\forall x \forall y \forall z_1 \forall z_2 \cdots \forall z_n (Exy \to (\alpha \to \alpha'))$$

where α is an atomic formula with variables included among $\{x, y, z_1, z_2, \ldots, z_n\}$, and α' is obtained from α by replacing some (possibly none, possibly some but not all, possibly all) occurrences of x by y. An example of a sentence of this form is

$$\forall x \forall y \forall z (Exy \to (Ezfx \to Ezfy)).$$

An example of a sentence *not* of this form is

$$\forall x \forall y \forall z (Exy \to (Ezfy \to Ezfx)).$$

Problem: Show that $E^{\mathfrak{A}}$ is an equivalence relation on the universe $|\mathfrak{A}|$, that $P^{\mathfrak{A}}$ induces a well-defined relation on equivalence classes, and that $f^{\mathfrak{A}}$ induces a well-defined function on equivalence classes.

Use your ordinary understanding of what it means for a sentence to be true in a structure. Do not try to use our formal definition of satisfaction. For example, it is fine to say that because the sentence $\forall x E x x$ is true in $\mathfrak{A}$, the relation $E^{\mathfrak{A}}$ is reflexive, and therefore the sentence $\forall y E f y f y$ is also true in $\mathfrak{A}$, or therefore $(f^{\mathfrak{A}}(d), f^{\mathfrak{A}}(d)) \in E^{\mathfrak{A}}$. You may also use your ordinary understanding of logical implication.

It is fine to introduce notation—for example, to define $a \equiv b$ to mean $(a, b) \in E^{\mathfrak{A}}$ (once you have shown $E^{\mathfrak{A}}$ is in fact an equivalence relation)—as long as you explain your notation.

Do be careful to distinguish between variable symbols and elements of the structure. Use w, x, y, z (possibly with indices) as variable symbols, and a, b, c, d (possibly with indices) as elements of $|\mathfrak{A}|$. This makes the distinction between the formal expression $\forall x$ and the phrase "for every a" immediately clear.

If you wish, when you are done, you can think about how you would rewrite your proof using the formal definition of satisfaction. However, that's not the point of this problem and I don't want to read those details.