Mathematics 69
Winter 2017
Homework Problems Assigned Monday, February 3

Some of what you are about to show will be used in the proof of the completeness theorem. For that reason, you may not use the completeness theorem in these problems. You may, however, use the soundness theorem.

(1.) Let $\mathcal{L}$ be a language for first-order logic, including the equality symbol (and any number of other symbols), and $\mathcal{L}^*$ be $\mathcal{L}$ without the equality symbol, but with an additional 2-place predicate symbol E. For any formula α of $\mathcal{L}$, define α^* to be the formula obtained from α by replacing every occurrence of $=$ with E. If Σ is a set of formulas of $\mathcal{L}$, define

$$\Sigma^* = \{\alpha^* \mid \alpha \in \Sigma\}.$$

(a.) For which logical axiom groups is it true that α is a logical axiom in that group iff α^* is an axiom in that group? An informal explanation of your answer is fine.

(b.) Which (if either) of the following are true? An informal explanation of why either one of these is true is fine. If either one of these is false, you should give a counterexample.

(i.) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction from Σ in the language $\mathcal{L}$, then $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ is a deduction from Σ^* in $\mathcal{L}^*$.

(ii.) If $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ is a deduction from Σ^* in the language $\mathcal{L}^*$, then $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a deduction from Σ in $\mathcal{L}$.

(c.) Show that if Σ is a consistent set of formulas in $\mathcal{L}$, then Σ^* is a consistent set of formulas in $\mathcal{L}^*$. You may use your answers to (a) and (b).

(d.) Show by example that it is possible for Σ to be inconsistent but Σ^* to be consistent. Be sure to explain why your example works.

(2.) Let $\mathcal{L}$ have equality, a one-place function symbol f, a two-place predicate symbol P, and no other parameters other than $\forall$. Suppose $\mathfrak{A}^*$ is a structure for $\mathcal{L}^*$ with the property that if α is a logical axiom of $\mathcal{L}$ in group 5 or group 6, then $\mathfrak{A}^*$ satisfies α^*. You showed in last week's homework that in this case, $E^{\mathfrak{A}^*}$ defines an equivalence relation on $|\mathfrak{A}^*|$, and $f^{\mathfrak{A}^*}$ and $P^{\mathfrak{A}^*}$ induce a well-defined function and a well-defined relation on equivalence classes.

Let $\mathfrak{B}$ be a structure for $\mathcal{L}$ (remember that E is not a symbol of $\mathcal{L}$) defined by setting

$$|\mathfrak{B}| = |\mathfrak{A}^*|/E^{\mathfrak{A}^*} = \{[a] \mid a \in |\mathfrak{A}^*|\},$$

where $[a]$ denotes the equivalence class of a under the equivalence relation $E^{\mathfrak{A}^*}$, and letting $f^{\mathfrak{B}}$ and $P^{\mathfrak{B}}$ be induced by $f^{\mathfrak{A}^*}$ and $P^{\mathfrak{A}^*}$. A a reminder, the induced function and relation are

$$f^{\mathfrak{B}}([a]) = [f^{\mathfrak{A}^*}(a)],$$

$$\langle [a], [b] \rangle \in P^{\mathfrak{B}} \iff \langle a, b \rangle \in P^{\mathfrak{A}^*}.$$

This second condition can be rewritten more informally, using infix notation for the relations translating the symbol P, as

$$[a]P^{\mathfrak{B}}[b] \iff aP^{\mathfrak{A}^*}b.$$

Let $\mathfrak{A}$ be the *reduct* of $\mathfrak{A}^*$ to a structure for $\mathcal{L}$. This means that

$$|\mathfrak{A}| = |\mathfrak{A}^*|,$$

$$f^{\mathfrak{A}} = f^{\mathfrak{A}^*},$$

$$P^{\mathfrak{A}} = P^{\mathfrak{A}^*}.$$

In other words, $\mathfrak{A}$ translates every parameter of $\mathcal{L}$ in exactly the same way as $\mathfrak{A}^*$. It merely has no translation for E, since E is not a symbol of $\mathcal{L}$.

Define a function h from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ by $h(a) = [a]$.

Notice that if s is a function from the set of variable symbols to $|\mathfrak{A}^*|$, then s is a variable assignment both for $\mathfrak{A}^*$ and for $\mathfrak{A}$, and $h \circ s$ is the variable assignment for $\mathfrak{B}$ defined by $(h \circ s)(v) = h(s(v)) = [s(v)]$.

(a.) Show that h is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

(b.) Show that if t is any term of $\mathcal{L}^*$ and s is a variable assignment for $\mathfrak{A}^*$, then

$$\overline{h \circ s}(t) = [\overline{s}(t)].$$

Use induction on terms.

(c.) Show that if α is any formula of $\mathcal{L}$ and s is a variable assignment for $\mathfrak{A}^*$, then

$$\mathfrak{A}^* \models \alpha^*[s] \iff \mathfrak{B} \models \alpha[h \circ s].$$

Use induction on formulas.