Mathematics 69

Winter 2017

Homework Problems Assigned Friday, February 10

For this assignment, $\mathcal{L}$ is the language of first-order logic with equality, countably many constant symbols c_0, c_1, $\dots$, c_n, $\dots$ and no other predicate, constant, or function symbols. We will find all the complete theories of $\mathcal{L}$.

Note, this isn't really six homework problems, it is one homework problem in 6 parts. You may use soundness, completeness, and compactness.

If $\mathfrak{A}$ is a structure for $\mathcal{L}$, define an equivalence relation on $\mathbb{N}$ by

$$m \equiv^{\mathfrak{A}} n \iff c_m^{\mathfrak{A}} = c_n^{\mathfrak{A}}.$$

(1.) Let $\equiv$ be any equivalence relation on $\mathbb{N}$. Show that there is a structure $\mathfrak{A}$ for $\mathcal{L}$ such that $\equiv^{\mathfrak{A}}$ is the same relation as $\equiv$. (Hint: Try letting the elements of $|\mathfrak{A}|$ be equivalence classes.)

(2.) Show that if $\mathfrak{A}$ is any infinite structure for $\mathcal{L}$, there is a countable structure $\mathfrak{B}$ such that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$, and in $\mathfrak{B}$, infinitely many elements are not named by constant symbols. In other words, we have that $\{b \in |\mathfrak{B}| \mid (\forall n)(b \neq c_n^{\mathfrak{B}})\}$ is infinite. (Hint: Use compactness.)

(3.) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures for $\mathcal{L}$ in which infinitely many elements are not named by constant symbols, and $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv_{\mathfrak{B}}$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(4.) Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two structures for $\mathcal{L}$, each of which is countable (or possibly finite). Which of the following conditions imply which others? In each case, explain why, or give a counterexample.

(a.) $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(b.) $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$.

(c.) $\equiv^{\mathfrak{A}}$ is the same relation as $\equiv^{\mathfrak{B}}$.

(5.) Suppose $\equiv$ is an equivalence relation on $\mathbb{N}$. Define

$$\Sigma_{\equiv} = \{c_n = c_m \mid m \equiv n\} \cup \{c_n \neq c_m \mid n \not\equiv m\}.$$

Show that if $\equiv$ has infinitely many equivalence classes, then $Cn\,\Sigma_{\equiv}$ is a complete theory. (Suggestion: Show that $\mathfrak{A}$ is a model of $Cn\,\Sigma_{\equiv}$ if and only if $\equiv_{\mathfrak{A}}$ is the same relation as $\equiv$. Use (2) and (3) to show that if $\mathfrak{A}$ and $\mathfrak{B}$ are any two models for $Cn\,\Sigma_{\equiv}$, then there are structures $\mathfrak{A}^*$ and $\mathfrak{B}^*$ elementarily equivalent to $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are isomorphic.)

(6.) Suppose $\equiv$ is an equivalence relation on $\mathbb{N}$ with finitely many equivalence classes. Describe all the complete (consistent) theories T with the property that $Cn\,\Sigma_{\equiv} \subset T$, by saying what sentences you need to add to $\Sigma_{\equiv}$ to produce a set of axioms for T. (Hint: Consider the possible ways to get finite or countable models of $Cn\,\Sigma_{\equiv}$ that are not isomorphic. Then consider whether these non-isomorphic structures have different theories.)

Since every complete theory for $\mathcal{L}$ contains some $Cn\,\Sigma_{\equiv}$, problems (5) and (6) together describe all the complete consistent theories of $\mathcal{L}$.