Lecture 4: Counting

At an ice cream shop, there are 31 flavors of ice cream and 5 toppings. There are also 2 types of cone. How many different two scoop ice cream cones can I make with one topping?

- 5 choices of topping
- 31 choices for top scoop
- 31 choices for bottom scoop
- 2 choices for cone

There are a total of $5 \cdot 31 \cdot 31 \cdot 2 = 9610$ different ice cream cones.

This example uses the <u>multiplication principle</u>.
If the are n ~~choices to~~ decisions to be made, with m_1 choices for the first decision, m_2 for the second decision, m_3 for the third decision, and so on, with m_n choices for the n^{th} decision, then there are

$$m_1 \cdot m_2 \cdot m_3 \cdots m_n$$

ways to ~~make the deci~~ choose the final outcome.

In the example, $n = 4$,
$m_1 = 5$, $m_2 = 31$, $m_3 = 31$, $m_4 = 2$
for a product of 9610.

The ice cream shop owners have decided to put out the ice cream in one row. How many ways are there to arrange the 31 flavors?

slot 1
slot 2
slot 3
⋮
slot 29
slot 30
slot 31

Say we fill slot 31 first. We have 31 flavors we can choose to put into this slot. Once we choose to put chocolate in slot 31, we only have 30 flavors to arrange in slots 1 through 30. If we fill slot 30 next, we have 30 choices. Suppose we pick vanilla. Now there are only 29 choices for slots 1 through 29. We continue in this manner until only 1 choice remains for slot 1.

Using the multiplication principle with $n = 31$, $m_1 = 31$, $m_2 = 30$, $m_3 = 29$, $\ldots$, $m_{31} = 1$ we get $(31)(29)(28)\cdots(3)(2)(1)$ possible arrangements of the ice cream.

The number $n(n-1)(n-2)\cdots(3)(2)(1)$ is denoted by $n!$, read "n factorial." This can be thought of as the number of (full) permutations on a set of n elements.

Suppose the ice cream shop owners only want to put out 5 flavors. Now, $n = 5$, and by the same argument as above, $m_1 = 31$, $m_2 = 30$, $m_3 = 29$, $m_4 = 28$, $m_5 = 27$.

By the multiplication principle, there are $(31)(30)(29)(28)(27) = 20{,}389{,}320$ ways to do this.

Frequently, sequences such as this without repetition will be used. We introduce the notation $P(n,k)$ to be the number of permutations of length k from an n-element set. It can be shown that $P(n,k) = \frac{n!}{(n-k)!}$

To see this in the example, multiply by

$\frac{26!}{26!}$ to get $\frac{31!}{26!}$.

Before Halloween, I purchase 5 bags of M&Ms and 3 bags of skittles. I cannot tell the difference between different bags of M&Ms. How many ways are there to arrange them on my shelf?

Supposing that I am just going to put them in a line, there are eight spots on my shelf and I just need to "pick" five of them to hold my M&Ms. The other three, by necessity, will hold the skittles.

If we pick a ~~permutation~~ length 5 permutation of the 8 slots, it would correspond to ordering the M&M's. (put the first bag in the first numbered slot, etc. (The first numberred slot refers to the first one in the permutation, not on my shelf.) ~~I don't care about the order of~~ ~~th~~ There are 5! ways to order the M&M's, and I don't care about that order, so the number is

$$\frac{P(8,5)}{5!} = \frac{8!}{3!\,5!}$$

When picking a subset from a set, the order does not matter. Such choices are called <u>combinations</u>. We denote the number of size k subsets from an n-element set by $\binom{n}{k}$, said "n choose k." It can be shown that $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

There are still questions on this, which will hopefully be cleared up in the next lecture.