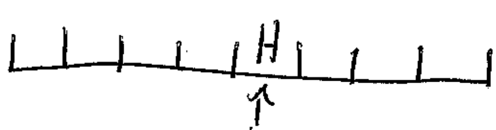
Lecture 6: Expected Value and Binomial Probability

In a binomial experiment, there are two possible outcomes, one of which is considered "good" and the other "bad." The standard experiment is tossing a coin. If we toss a coin 8 times, what is the probability of seeing exactly 0, 1, 4, or 8 heads?

The probability of seeing heads with one toss is exactly $\frac{1}{2}$. Using the multiplication principle, we see that there ~~are~~ is a probability of $(\frac{1}{2})^8$ of not seeing heads at every trial.

Seeing exactly one head is slightly more complicated. We can think of the trials as consecutive slots and pick where our successes will occur.



There are $\binom{8}{1}$ ways to select the slots for successes. Once we have decided the location of the heads, there is a probability of $\frac{1}{2}$ that the outcome of those trials are heads, and a probability of $\frac{1}{2}$ that the outcome of any other trial is not heads, so the probability is $\binom{8}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^7$ that exactly 1 outcome will be heads.

Similarly, for 4 heads, the probability is $\binom{8}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4$ and for 8 heads it is $\binom{8}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0$.

Suppose, instead of a standard coin we toss a coin which is weighted so that the probability of heads is $3/4$. Now what is the probability of seeing exactly 0, 1, 4 and 8 heads?
The analysis here is similar. To see 0 heads, all outcomes must be tails, and in any one trial this occurs with

probability $\frac{1}{4}$. The total probability is $\left(\frac{1}{4}\right)^8$ of seeing 0 heads.

To see 1 heads, we pick where it is going to occur, there are $\binom{8}{1}$ ways to do this, and get the probability that exactly that slot is heads. $\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^7$. The final probability is $\binom{8}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^7$.

For 4 heads, the probability is $\binom{8}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^4$, and for 8 heads, the probability is $\binom{8}{8}\left(\frac{3}{4}\right)^8\left(\frac{1}{4}\right)^0$.

This is generalized to the following theorem:

<u>Thm</u>: For an experiment with probability p of success which is to be performed n times, the probability of seeing exactly k successes is $\binom{n}{k} p^k (1-p)^{n-k}$.

Suppose I am buying buttons, which come in packages of five. The manufacturing process produces bad buttons with a probability of .01. What is the probability that all the buttons in the package I buy are good buttons?

$$\binom{5}{0}(.99)^5(.0\not{1})^0.$$

I need to replace 100 buttons on a dress. How many packages of buttons should I buy?

If I buy 20 packages, there is a high probability that at least one is bad: $1 - p(\text{all good}) = 1 - \binom{100}{0}(.99)^{100} = .63$

so I should buy 21 packages.

21 packages will be enough with probability

$$\binom{105}{0}(.99)^{105} + \binom{105}{1}(.99)^{104}(.01)^1 + \binom{105}{2}(.99)^{103}(.01)^2 + \binom{100}{3}(.99)^{102}(.01)^3 + \binom{105}{4}(.99)^{101}(.04)^4 + \binom{105}{5}(.99)^{100}(.04)^5$$

= pretty good.

But one way to look at this is through expected value.

We define a random variable X to be something that takes on different nonnegitive real values.

Some examples of random variables include the number of heads in 10 coin tosses, the number of people in a sample who have a college education, etc.

One useful random variable is an indicator random variable. For any event E, if we pick an outcome x we say

$$X = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

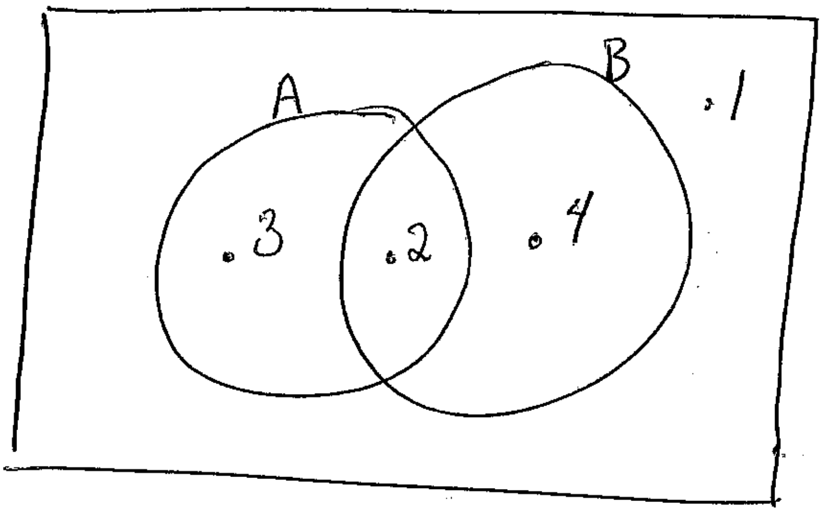
If the event were "in class today" from the sample space of "students in this class." and I pick a student at random, $X = 1$ for any of you here and $X = 0$ for any student that did not show up today.

This allows us to define an expected value.

Suppose X is a random variable which takes on values $x_1, x_2, \ldots, x_n$. We define the expected value of X to be

$$E[X] = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n).$$

For example, let A and B be two events as in the following Venn Diagram:



Let X be an indicator random variable for A and Y an indicator random variable for B. What is $E[X+Y]$?

$X+Y$ can take on 3 values, 0, 1, and 2.

$X+Y=0$ with probability .1.

$X+Y=2$ ~~when~~ for outcomes in $A \cap B$, which has probability .2.

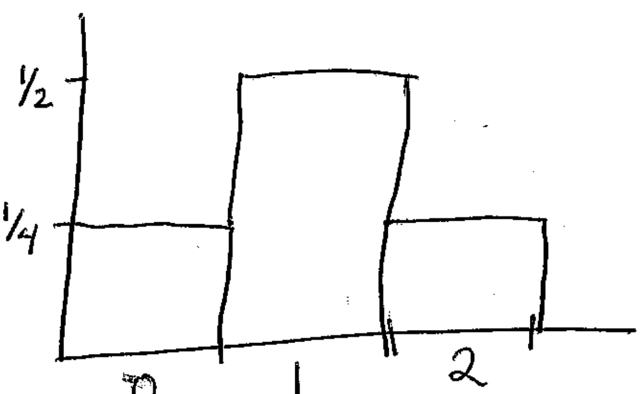
$X+Y=1$ for the remaining outcomes, so has probability .7.

$$E[X+Y] = (0)p(X+Y=0) + (1)p(X+Y=1) + (2)p(X+Y=2)$$
$$= (0)(.1) + (1)(.7) + (2)(.2)$$
$$= 1.1$$

It can be shown that for a binomial experiment with n trials and probability p of success, the expected number of successes is np.

We've introduced Venn Diagrams and trees to organize probability. Another organizational tool is a histogram. Here, the outcomes are graphed verses their probability. A bar represents each outcome.

For tossing two coins, here is a histogram for the number of heads:



For tossing five coins: