April 22 - Lecture 11

Annuities

An <u>annuity</u> is a sequence of equal payments made at equal periods of time.

If the payments are made with the same frequency as compounding, it is an <u>ordinary annuity</u>.

Most of the time, the easiest form of an annuity is a retirement fund.

<u>Example 1</u>: At the age of fifty-five, you suddenly realize that in ten years you want to retire. You begin depositing \$2000 into an account which pays 8% interest, compounded annually. How much will the annuity be worth if you continue depositing \$2000 annually?

Look at the ammounts deposited ~~#~~ as if they were deposited into different accounts. Your \$2000 this year will earn interest 9 times, next year's will earn interest eight times, and the last \$2000 will not earn any interest. The final amount is

$$(2000)(1.08)^9 + (2000)(1.08)^8 + \cdots + (2000)(1.08)^0$$

This sum looks scary, but it's a very common kind of sum, so there is a formula for it.

A <u>geometric sequence</u> is a sum of the form

$$x + xr + xr^2 + \cdots + xr^n$$
$$= \sum_{i=0}^{n} xr^i$$

r is the common ratio between consecutive terms in the sequence.

<u>Thm</u>: $\displaystyle\sum_{i=0}^{n} xr^i = x \, \frac{1-r^{n+1}}{1-r}$

<u>Proof</u>: Let $S = \displaystyle\sum_{i=0}^{n} xr^i$.

Consider ~~[illegible]~~ $S - rS = S(1-r)$

$$\begin{aligned} & x + xr + xr^2 + \cdots + xr^n \\ - \big(& \quad rx + rxr + \cdots + rxr^{n-1} + rxr^n \big) \\ = \; & x - xr^{n+1} = x(1-r^{n+1}) \end{aligned}$$

So $S(1-r) = x(1-r^{n+1})$

and $S = x \, \dfrac{1-r^{n+1}}{1-r}$ as desired.

This allows us to write the sum from example 1 as

$$\sum_{i=0}^{9} 2000(1.08)^i = 2000\left(\frac{1-(1.08)^{10}}{1-1.08}\right)$$
$$= \boxed{\$28,973.12}$$

In general, for an account with an interest rate r, ~~receiving~~ compounded m times a year, and receiving a payment of R dollars in each period, the value after t years is

$$R\,\frac{1-\left(1+\frac{r}{m}\right)^{mt}}{-\frac{r}{m}} = -R\left(\frac{m}{r}\right)\left(1-\left(1+\frac{r}{m}\right)^{mt}\right)$$

Annuities which recieve payments, such as these, are called <u>sinking funds</u>

For all of these, it is assumed that payments are made at the end of the compounding periods

For payments which are made at the beginning of the compounding period, the formula is

$$-R\left(\frac{1-(1+i)^{n+1}}{i}\right) - R$$

where $i=\frac{r}{m}$ and n is the number of periods in which payments are made. This is called an <u>annuity due</u>.

This lecture was short. ~~Tomorrow~~: Monday:
Car payments.
It is recommended that you use the rest of the class period to work on your projects or the weekend homework.

Homework #10

5.1 55, 59, 60
5.2 25, 27, 50, 60-63, 67