Math 71

Homework 5

1. As usual, for positive integers, let $[m, n]$ denote the least common multiple of m and n, and (m, n) their gcd.

   (a) **Using the First Isomorphism Theorem** show that there is an injective homomorphism $\mathbb{Z}/[m,n]\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Note: to use the first isomorphism theorem means you do **not** (directly) define a homomorphism $\mathbb{Z}/[m,n]\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

   (b) Show that the above mapping is surjective (and hence an isomorphism) if and only if $(m, n) = 1$, in which case $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

2. Let $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$ be finite cyclic groups of order m and n respectively. Show that $G_1 \times G_2$ is cyclic if and only if $(m, n) = 1$. This should follow easily from the previous problem.

3. Let $\varphi : \mathbb{Z}/18\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ be the homomorphism satisfying $\varphi(\bar{1}) = \bar{10}$, that is the class of 1 in $\mathbb{Z}/18\mathbb{Z}$ is mapped to the class of 10 in $\mathbb{Z}/12\mathbb{Z}$.

   (a) Show that φ is well-defined, but the map $\psi : \mathbb{Z}/18\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ induced by $\psi(\bar{1}) = \bar{3}$ is not.

   (b) Compute the kernel of φ, and apply the first isomorphism theorem.

   (c) To what well-known group is $Im(\varphi)$ isomorphic? Using this representation, your statement invoking the first isomorphism theorem should look like an application of the third isomorphism theorem; make sure it does!

4. Let $\mathbb{Z}[x]$ denote the set of polynomials with coefficients in $\mathbb{Z}$. It is obvious that $\mathbb{Z}[x]$ is an additive group . The example will be even more important as we start the study of rings.

   (a) Define $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ by $\varphi(f) = f(0)$. Show that φ is a surjective homomorphism and compute its kernel. What statement does the first isomorphism theorem yield?

   (b) Do the same as above with $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $\varphi(f) = f(3)$.

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