Math 71 Fall 2015
Homework #3: symmetric groups, group actions

(1) Recall that $\mathfrak{A}_n$ denotes the alternating group, that is the subgroup of even permutations in $\mathfrak{S}_n$.

(a) Assume $n \geq 3$. Prove that $\mathfrak{A}_n$ is generated by 3-cycles.

Let $p \geq 5$ be a prime number and H a subgroup of $\mathfrak{S}_p$ such that $[\mathfrak{S}_p : H] \leq p - 1$.

(b) Prove that H contains the p-cycles.

(c) Prove that $[\mathfrak{S}_p : H] \in \{1, 2\}$.

(d) Deduce from this another concrete example showing that there is no general converse to Lagrange's Theorem.

(2) Let G be a finite group, and H a subgroup. For $g \in G$, and $\bar{x} \in G/H$, let $\sigma_g(\bar{x}) = \overline{gx}$.

(a) Verify that σ_g is a well-defined map from G/H to itself.

(b) Prove that the map $\sigma : g \mapsto \sigma_g$ is a group homomorphism from G to $\mathrm{Bij}(G/H)$.

(c) Assume that $[G : H] = p$, where p is the smallest prime factor of $\#G$.
Prove that H is normal in G.

(3) Let G be a finite group acting on a finite set X.

(a) Assume that every orbit contains at least 2 elements, that $\#G = 15$ and that $\#X = 17$. Determine the number of orbits and the cardinal of each of them.

(b) Assume that $\#G = 33$ and $\#X = 19$. Prove that the action has a fixed point.

(4) Let G be a group and H a subgroup.

(a) Recall how G acts on G/H by left multiplication and determine the stabilizer of the coset gH.

If G acts on two sets X and Y, a map f from X to Y is said G-*equivariant* if it satisfies

$$f(g \cdot x) = g \cdot f(x)$$

for every $g \in G$ and $x \in X$.

(b) Prove that the inverse of a G-equivariant bijection is G-equivariant.

(c) Assume that G acts transitively on X, let $x \in X$ and $H = \mathrm{Stab}_G(x)$. Construct a G-equivariant bijection between X and G/H.

(d) Let H and K be subgroups of G. Prove that there exists a G-equivariant map from G/H to G/K if and only if H is contained in a conjugate of K, and that such a map is necessarily surjective.

(e) Prove that there exists a G-equivariant bijection between G/H and G/K if and only if H and K are conjugate in G.